Simple maps (also) converge to the Brownian map

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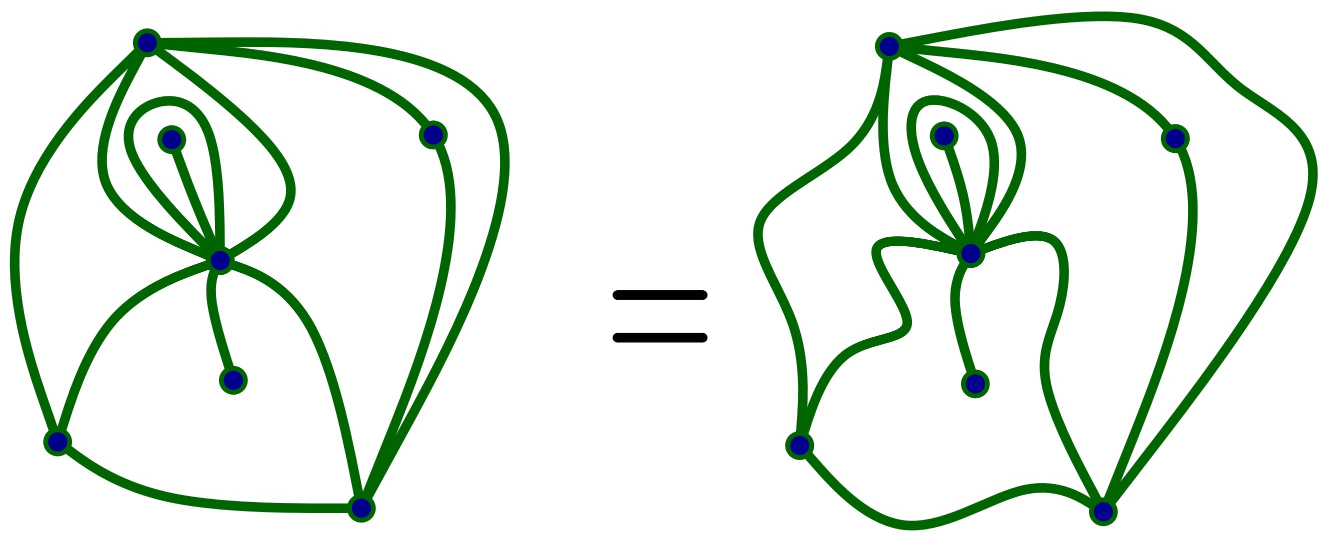
joint work with Olivier Bernardi (Brandeis University),
Gwendal Collet and Eric Fusy (LIX)

MAC2 Workshop,
7th July 2014

Planar Maps – Definition.

A **planar map** is the proper embedding of a planar connected graph in the 2-dimensional sphere seen up to continuous deformations.



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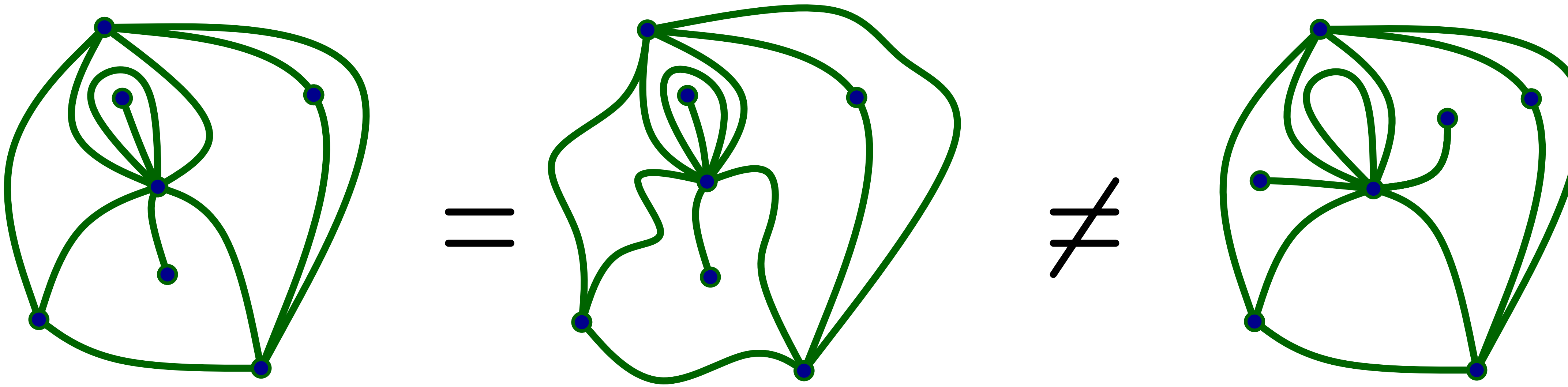

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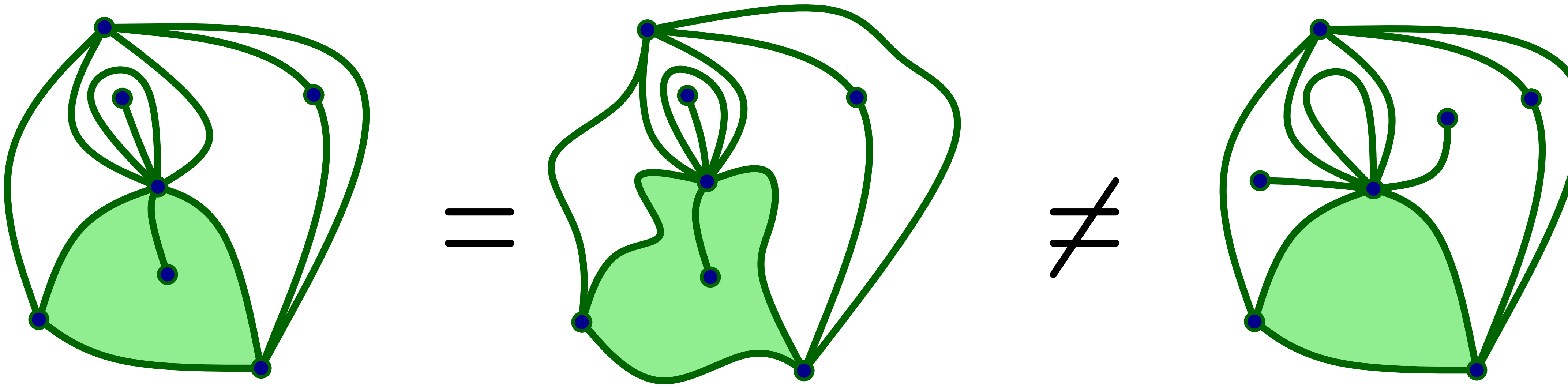

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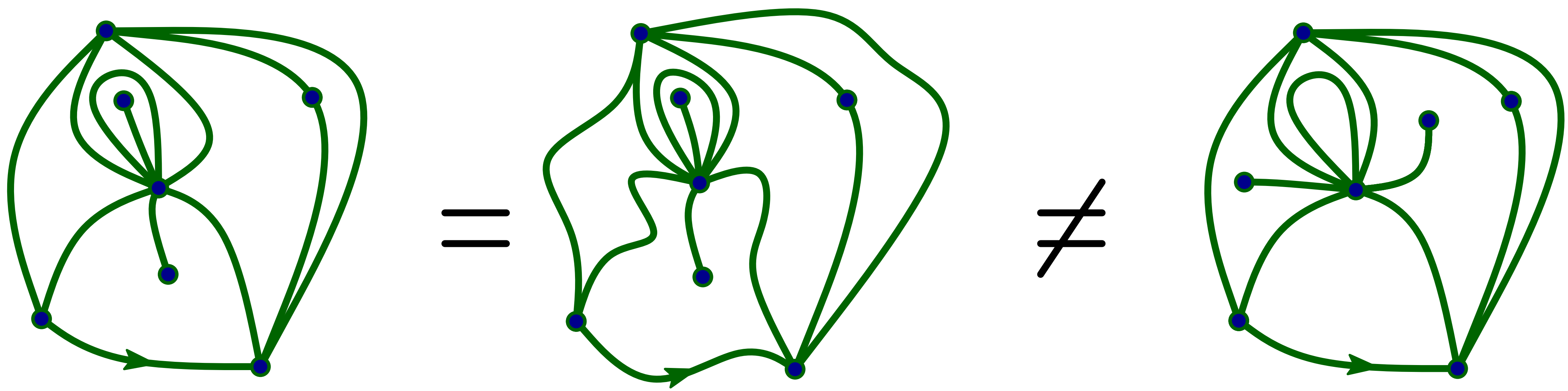

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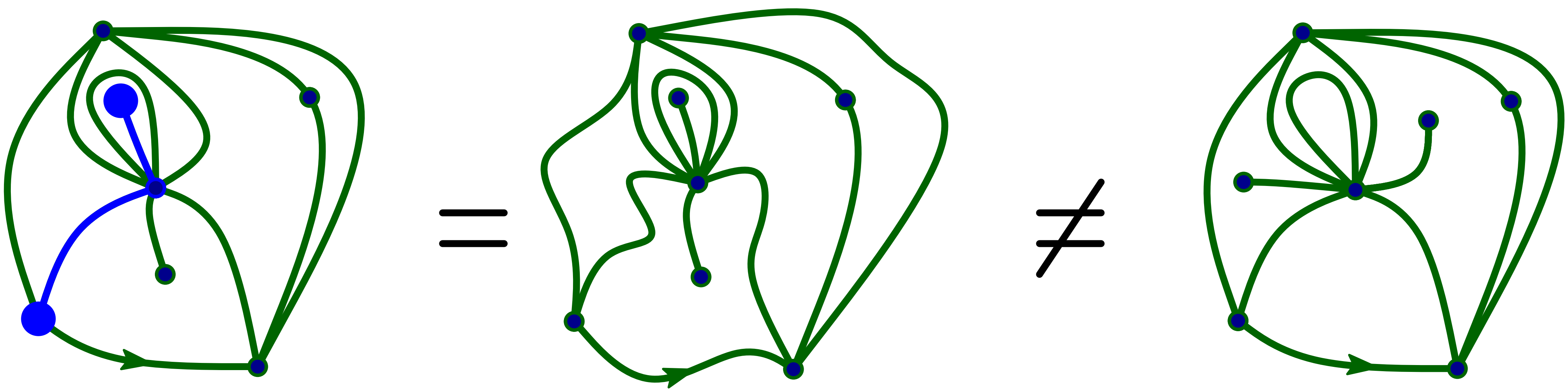

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Plane maps are **rooted** : by orienting an edge.

Distance between two vertices = number of edges between them.
Planar map = **Metric space**

Why maps ?

What the motivation for studying maps instead of graphs ?

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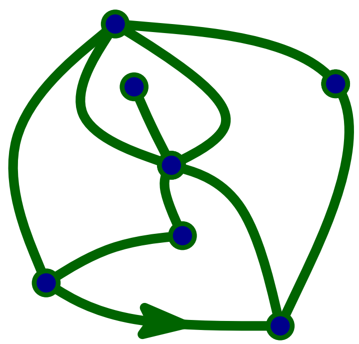
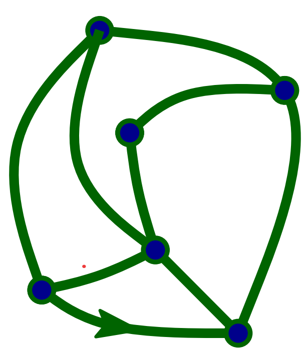
Euler Formula : $\#\,\text{vertices} + \#\,\text{faces} = \mathbf{2} + \#\,\text{edges}$

A quadrangulation with n faces has $2n$ edges and $n+2$ vertices.

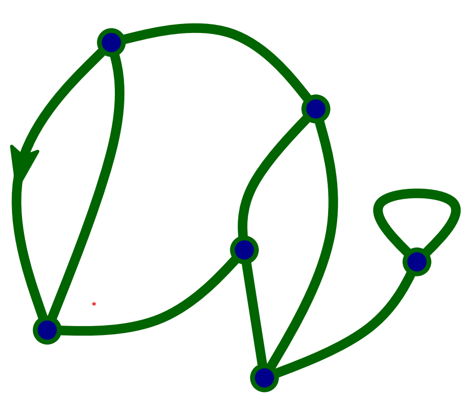
Which maps ?



Quadrangulations (all faces have degree 4)



Simple maps (no loops nor multiple edges)



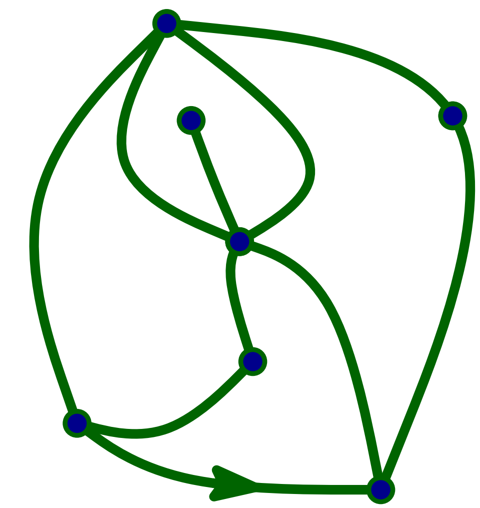
Cubic maps (all vertices have degree 3)

Random maps

$\mathcal{Q}_n = \{$Quadrangulations of size $n\}$
$= n+2$ vertices, n faces, $2n$ edges

$Q_n =$ Random element of $\mathcal{Q}_n$

$(V(Q_n), d_{gr})$ is a random compact metric space



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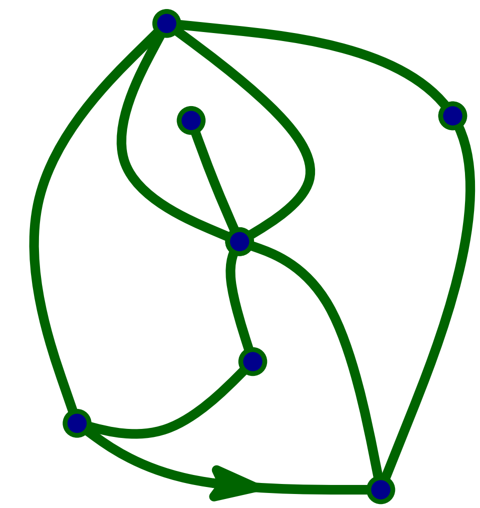
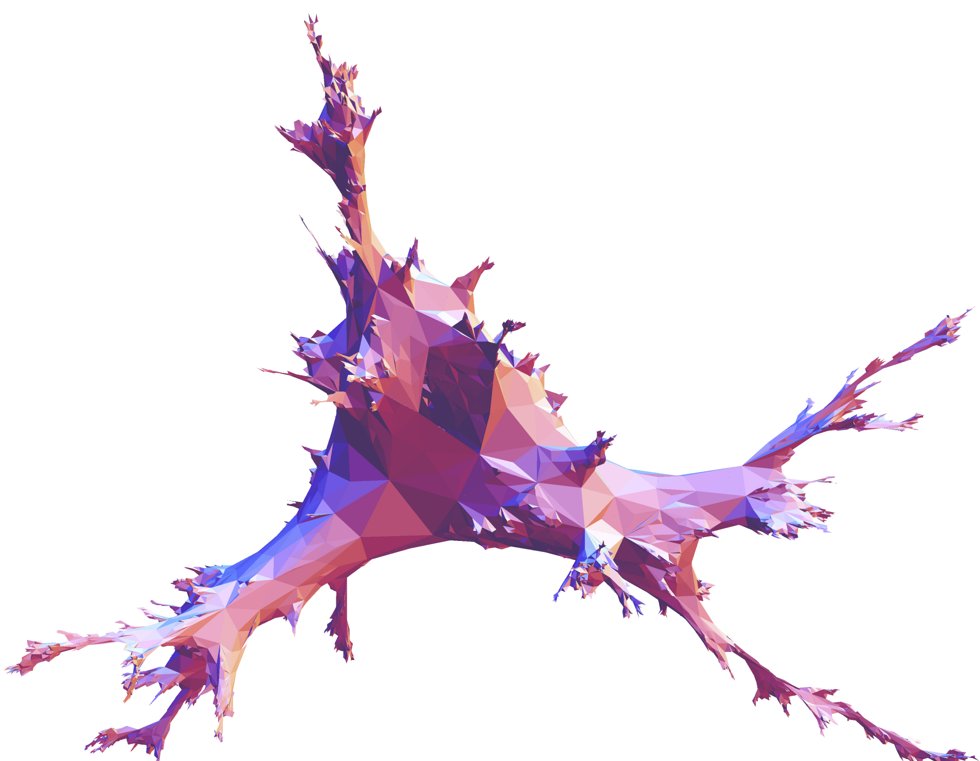


Simulations by N.Curien

Random maps

What is the behavior of Q_n when n goes to infinity ?
typical distances?
convergence towards a continuous object ?



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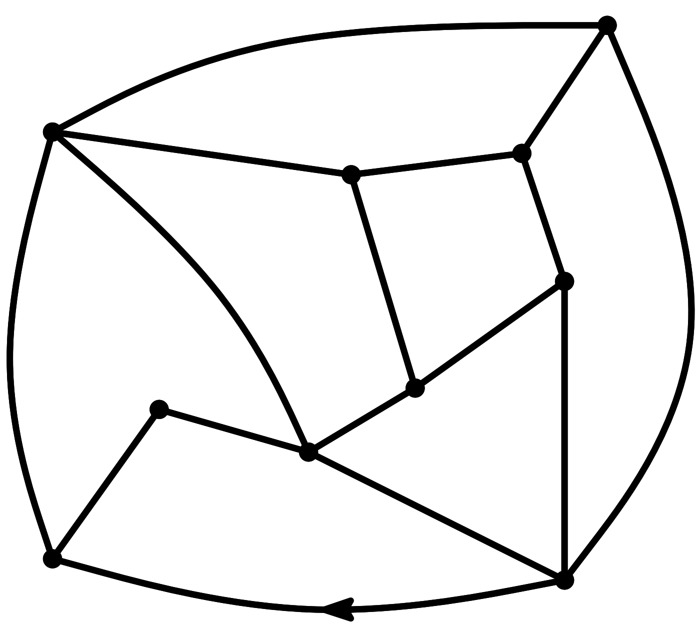

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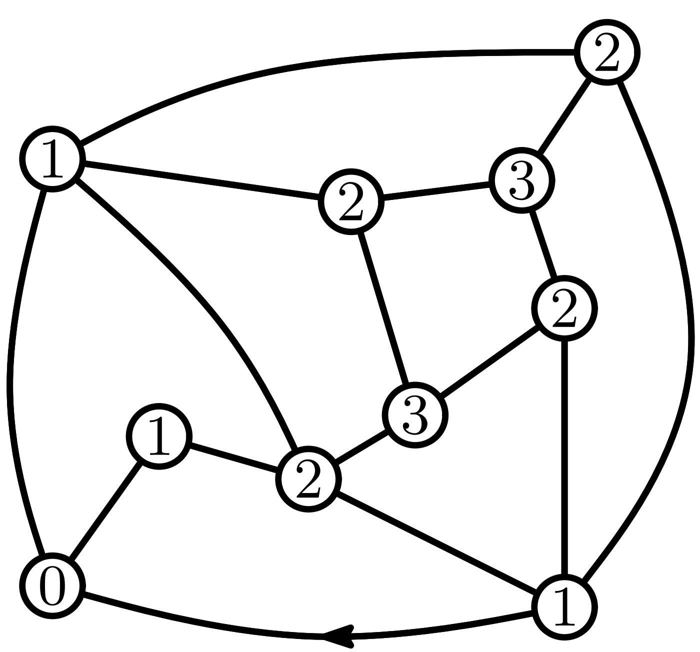

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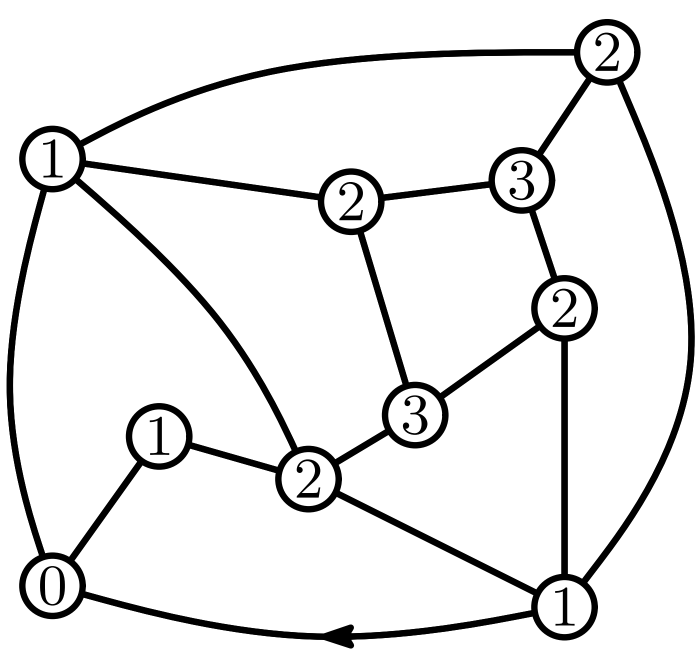

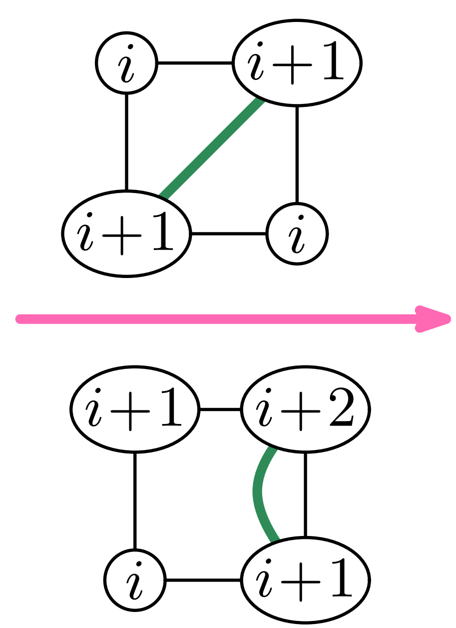

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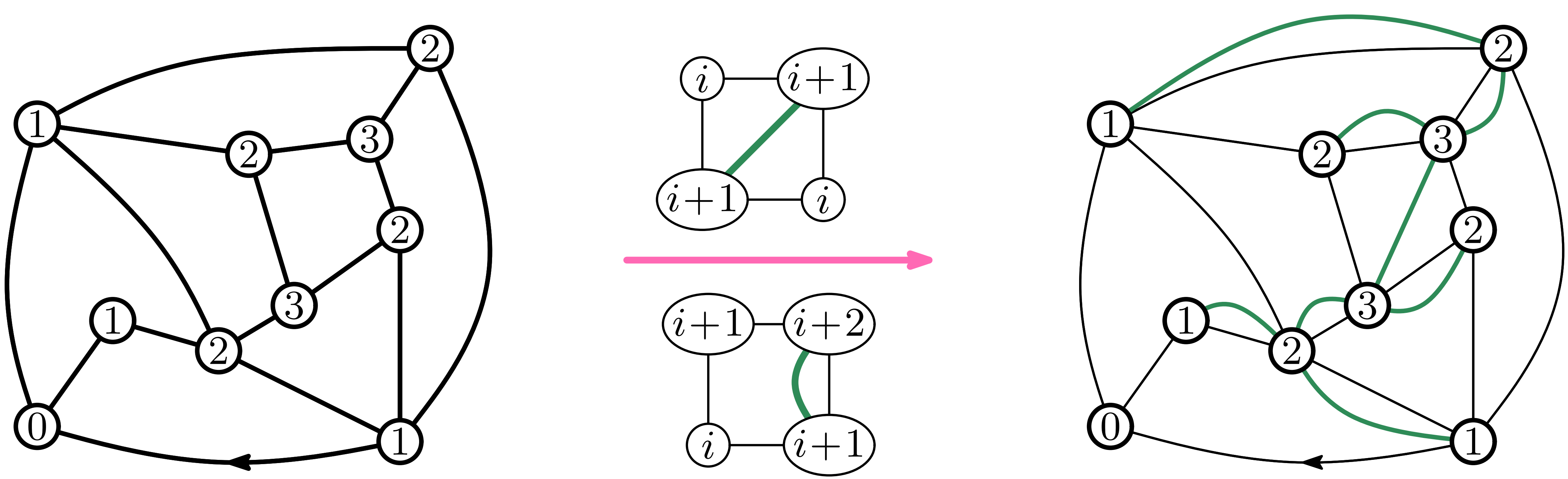

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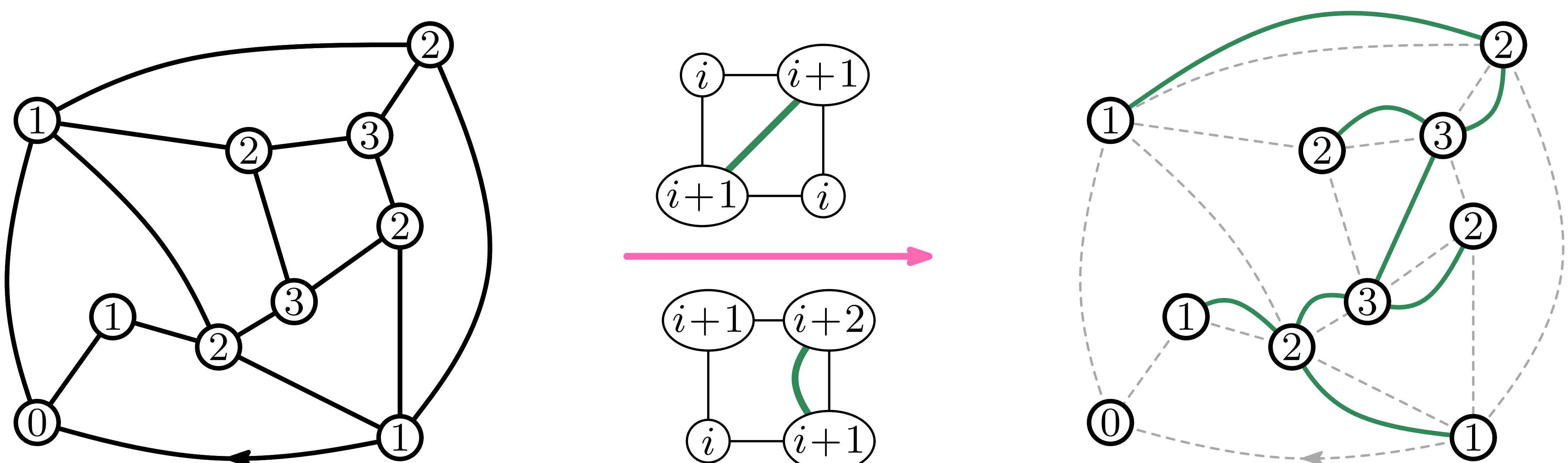

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• distance between two random points $\sim n^{1/4}$ + law of the distance
[Chassaing-Schaeffer '04]

• cvgence of normalized quadrangulations + properties of the limit
[Marckert-Mokkadem '06], [Le Gall '07], [Le Gall, Paulin '08] [Miermont '08]

Hausdorff dimension $= 4$ ([Le Gall '07])

topology of the limit $=$ sphere ([Le Gall, Paulin '08], [Miermont '08])

• cvgence of normalized quadrangulations towards the **Brownian map** for Gromov-Hausdorff topology, [Miermont '13], [Le Gall '13]

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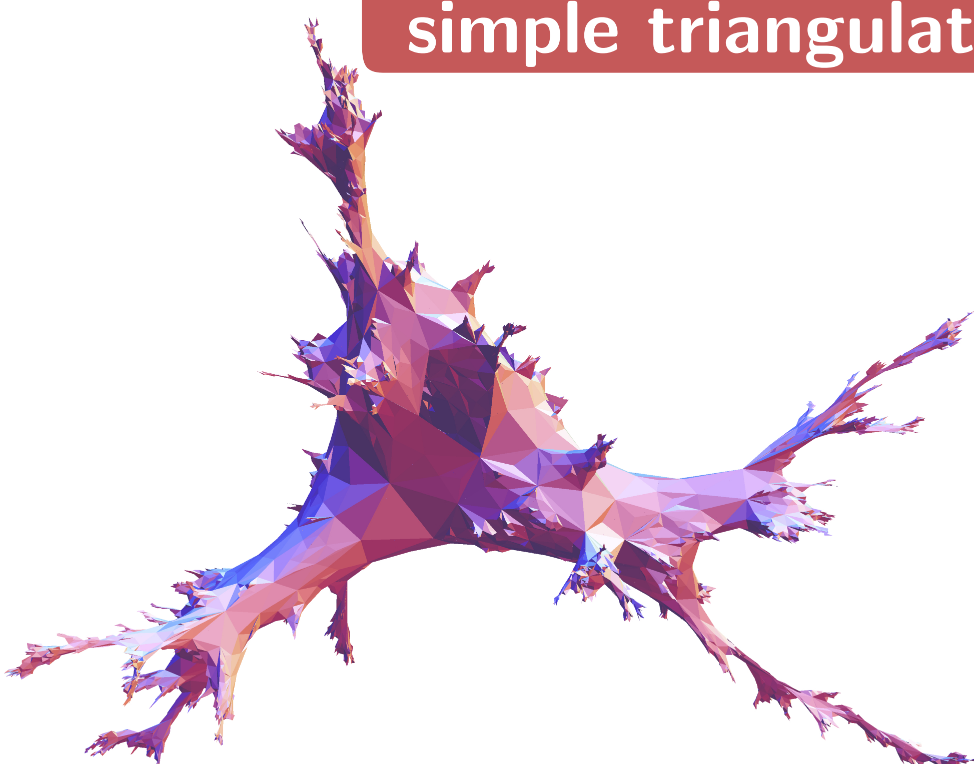

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All "reasonable models" of maps (properly rescaled) are expected to converge towards it.

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- $2p$-angulations and triangulations [Le Gall, '13]
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Today:
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An important remark:

Thanks to an argument of [Le Gall '13], enough to :

- understand the distance between any point and the root,
- show that the distance between two points is tight.
- prove the invariance under rerooting

and use the result of [Miermont '13], [Le Gall '13] to conclude.

So far: no **direct proof** known.

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- encode the maps by some trees,
- **study the limits of trees**,
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Theorem : [A., Bernardi, Collet, Fusy]
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- The Brownian Map

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[Schaeffer '98]
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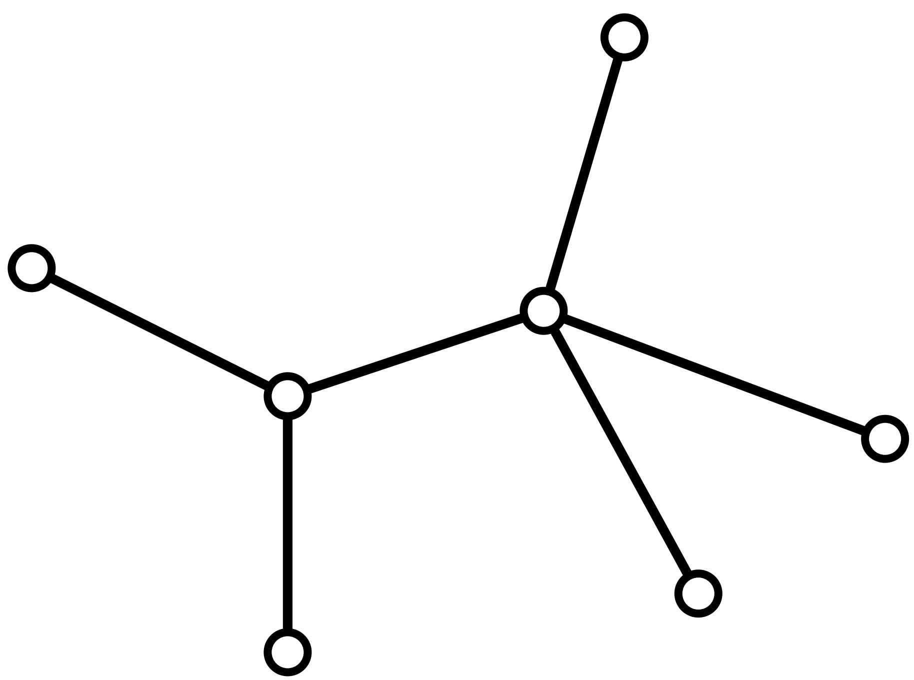
$$\# \text{ closing stems} = \# \text{ opening stems}$$



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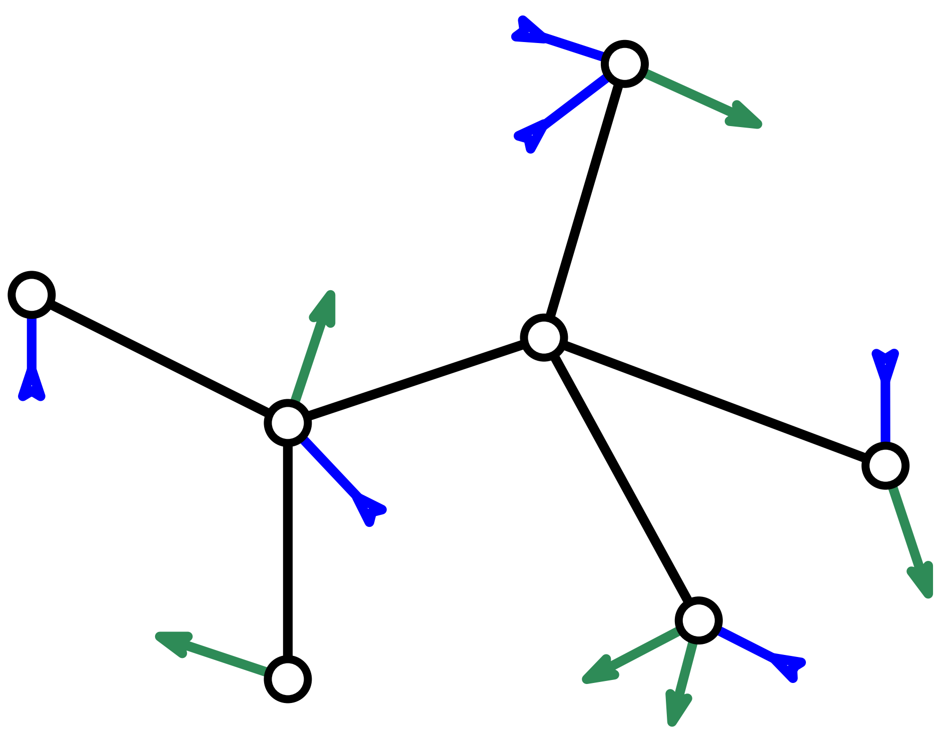

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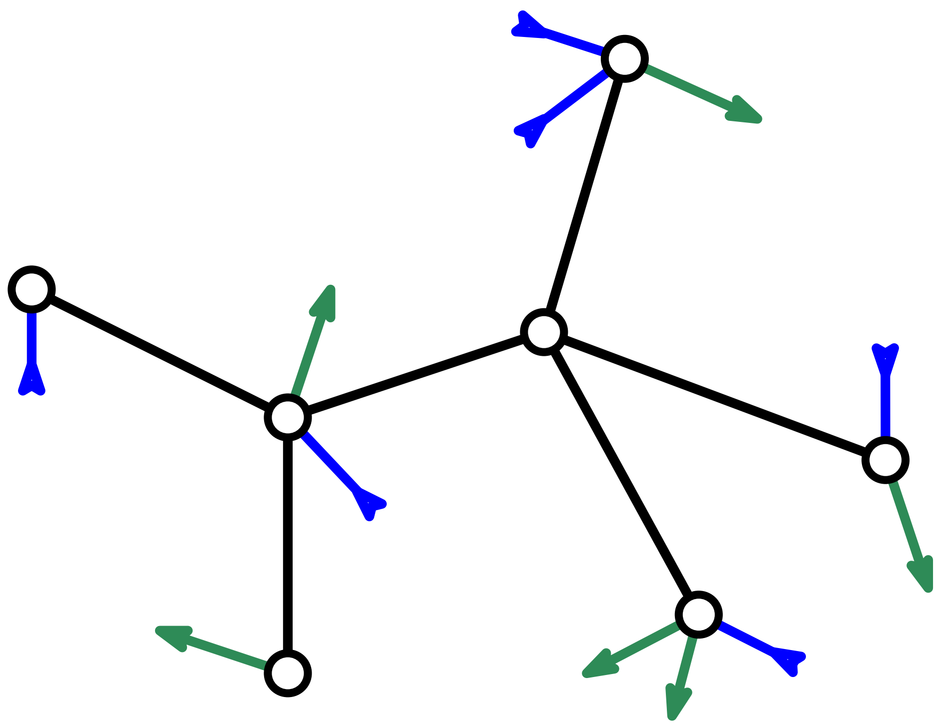

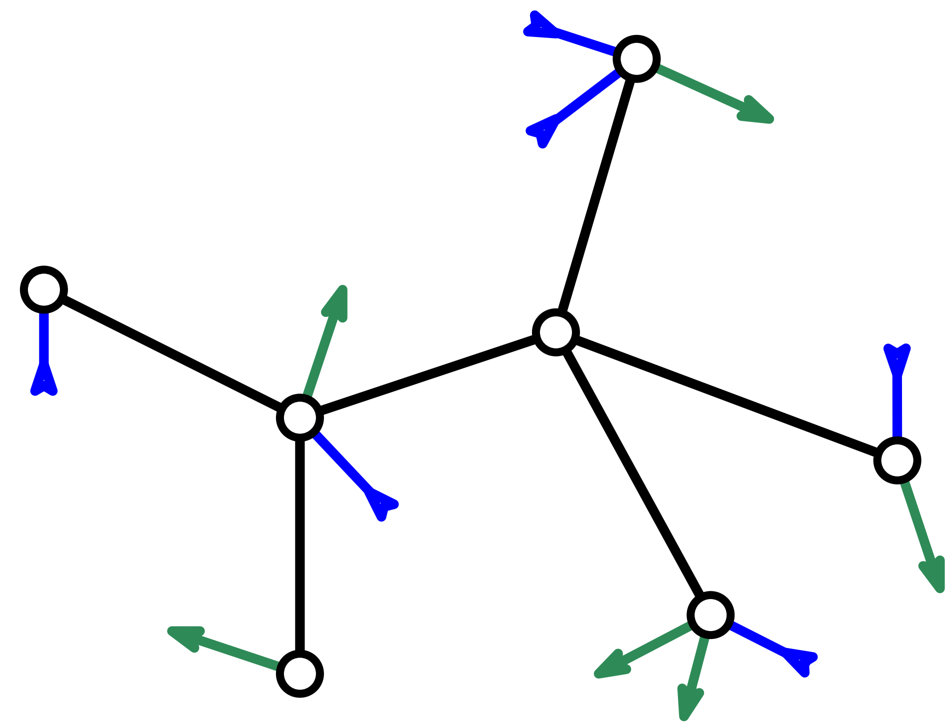

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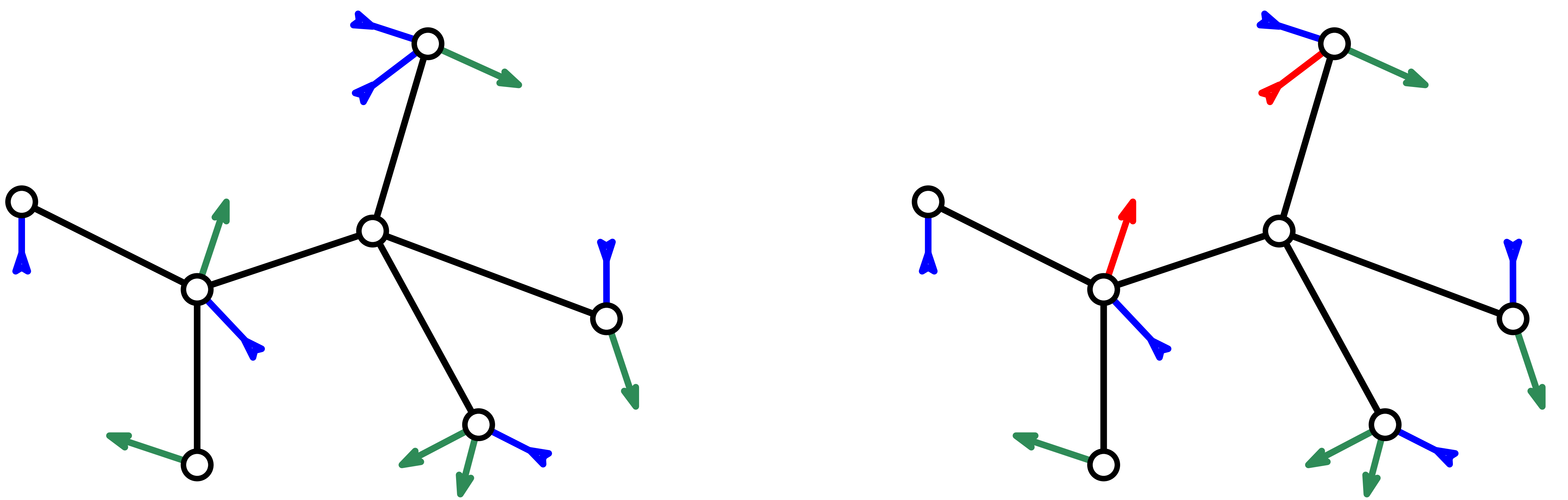

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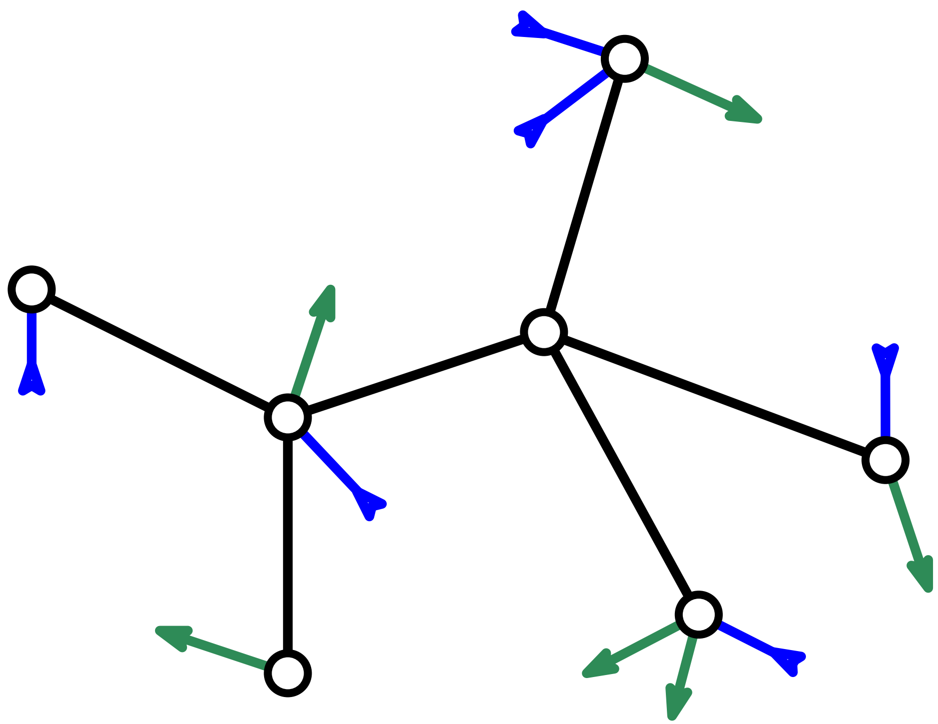

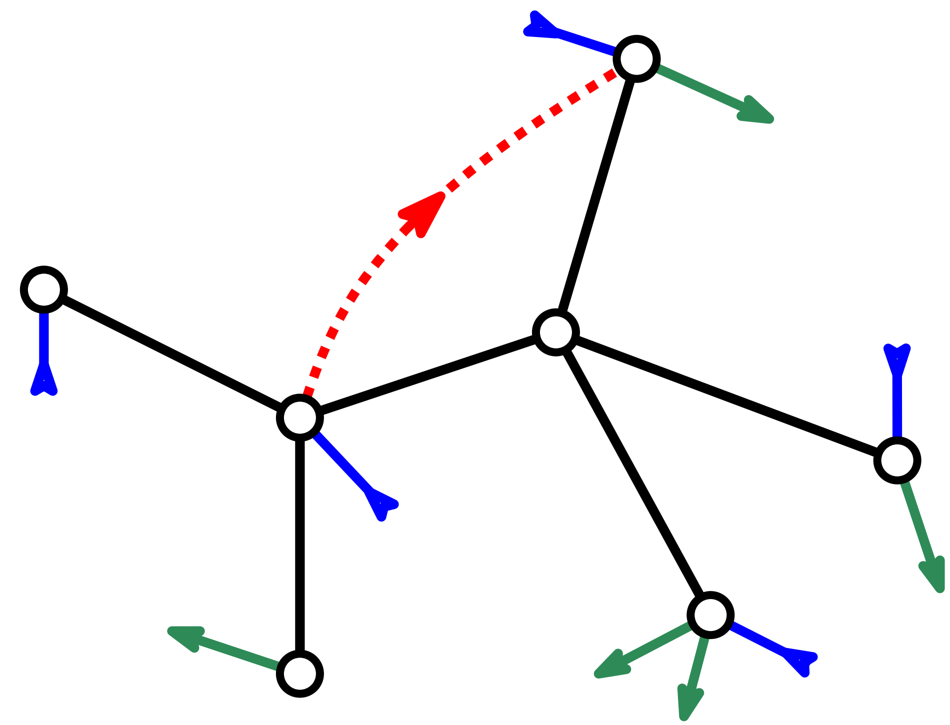

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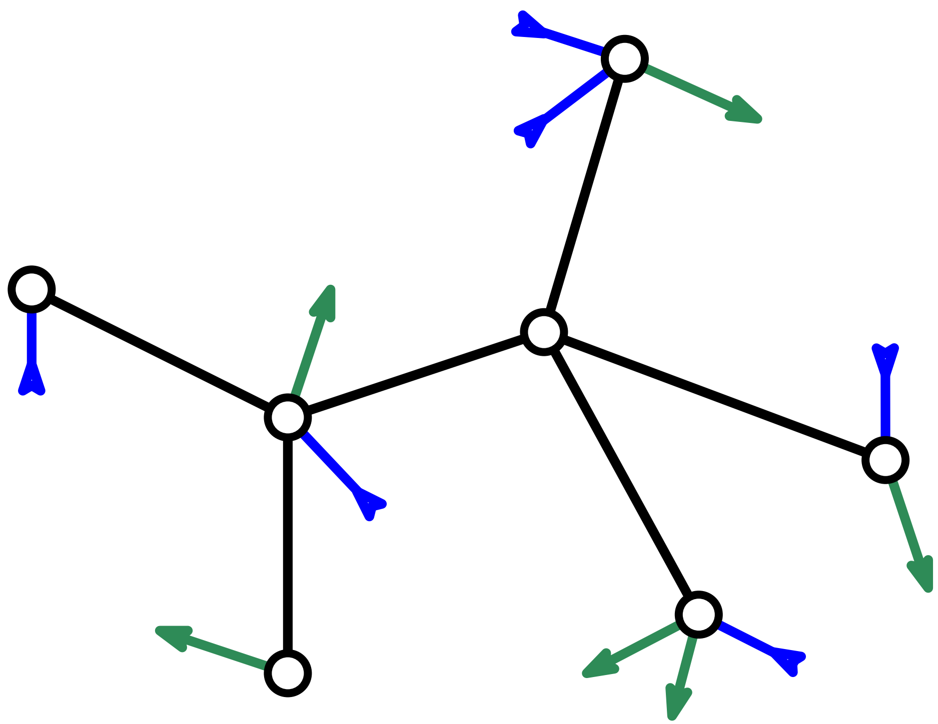

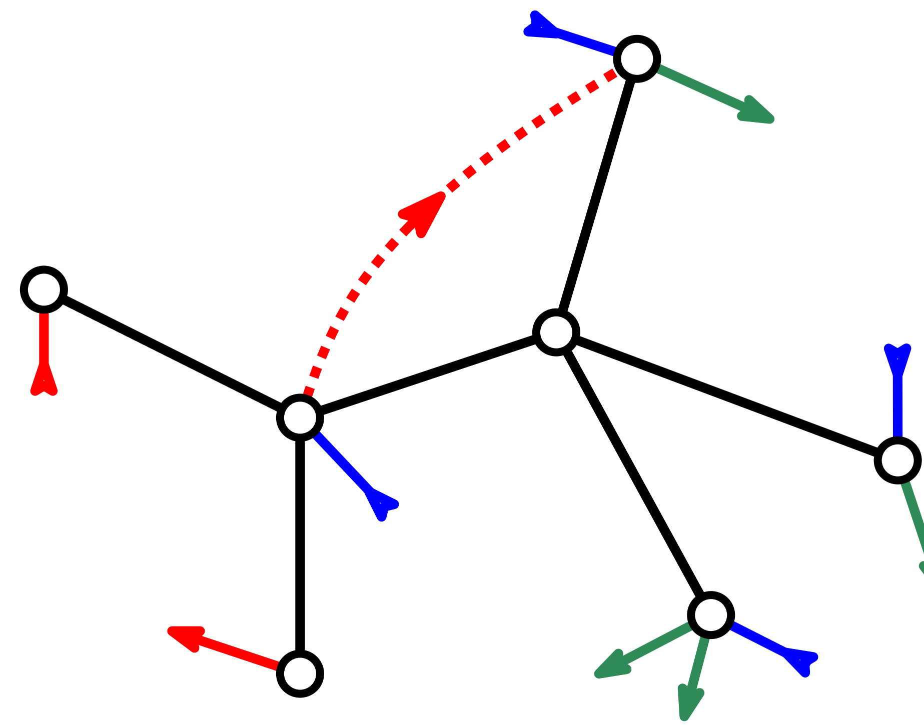

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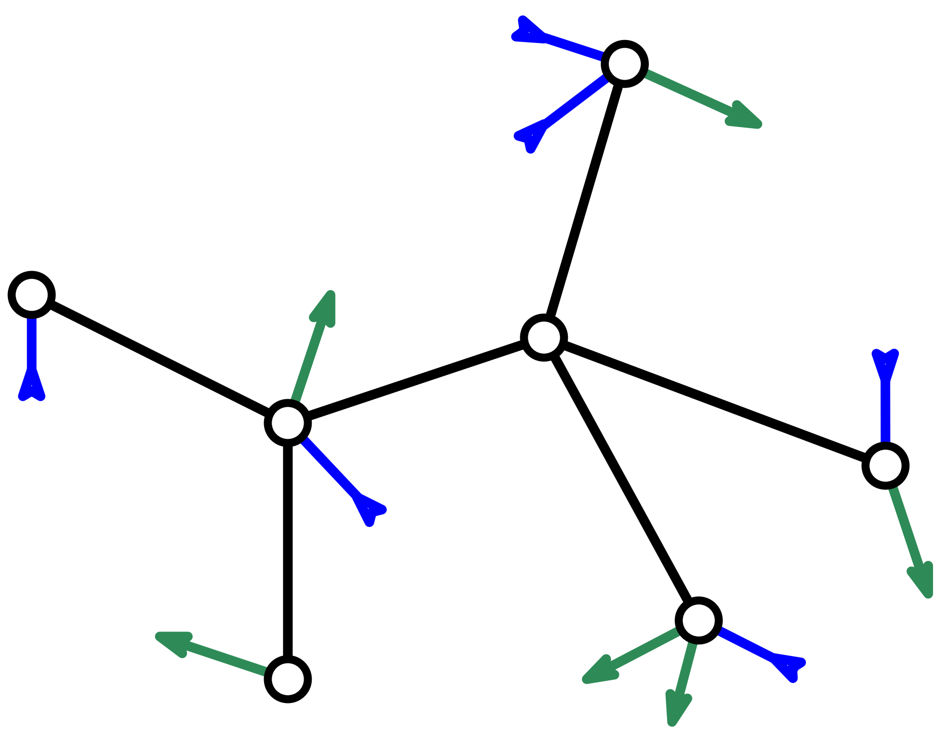

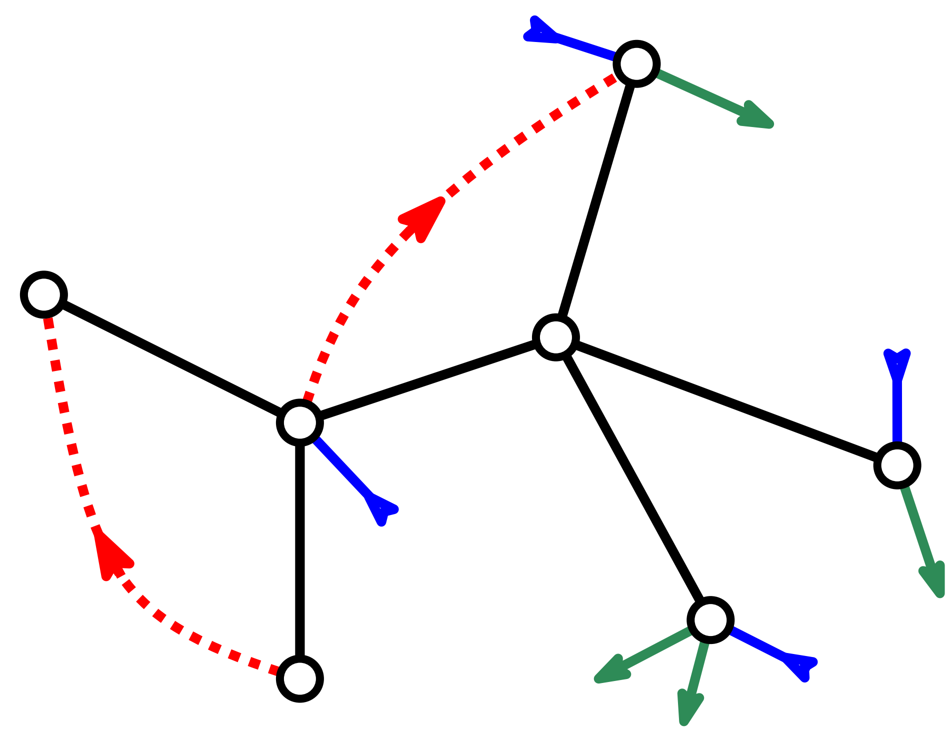

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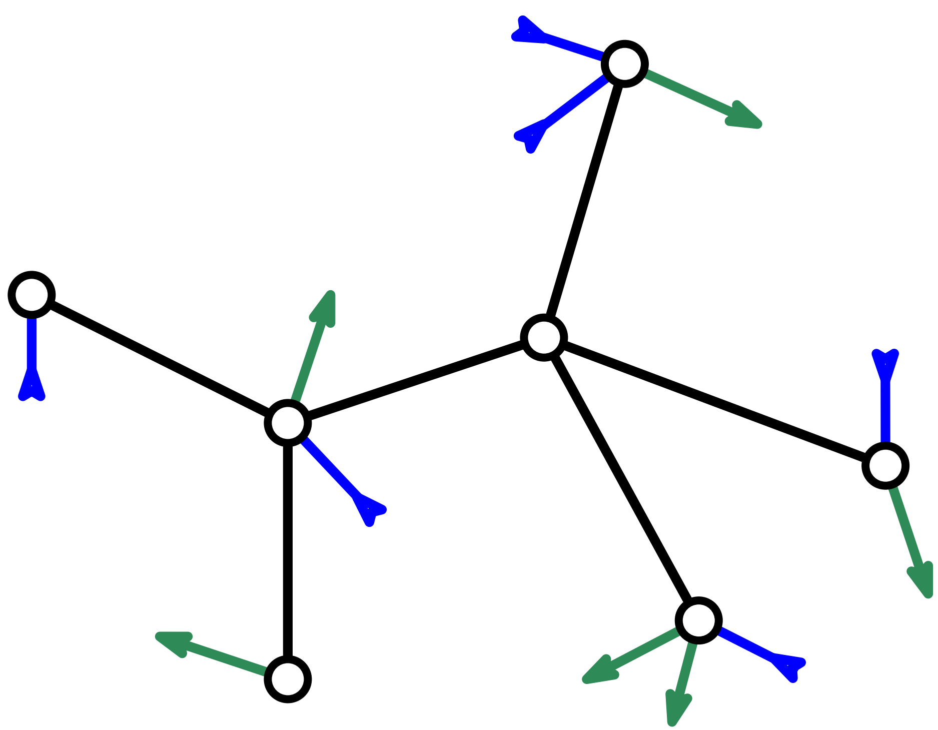

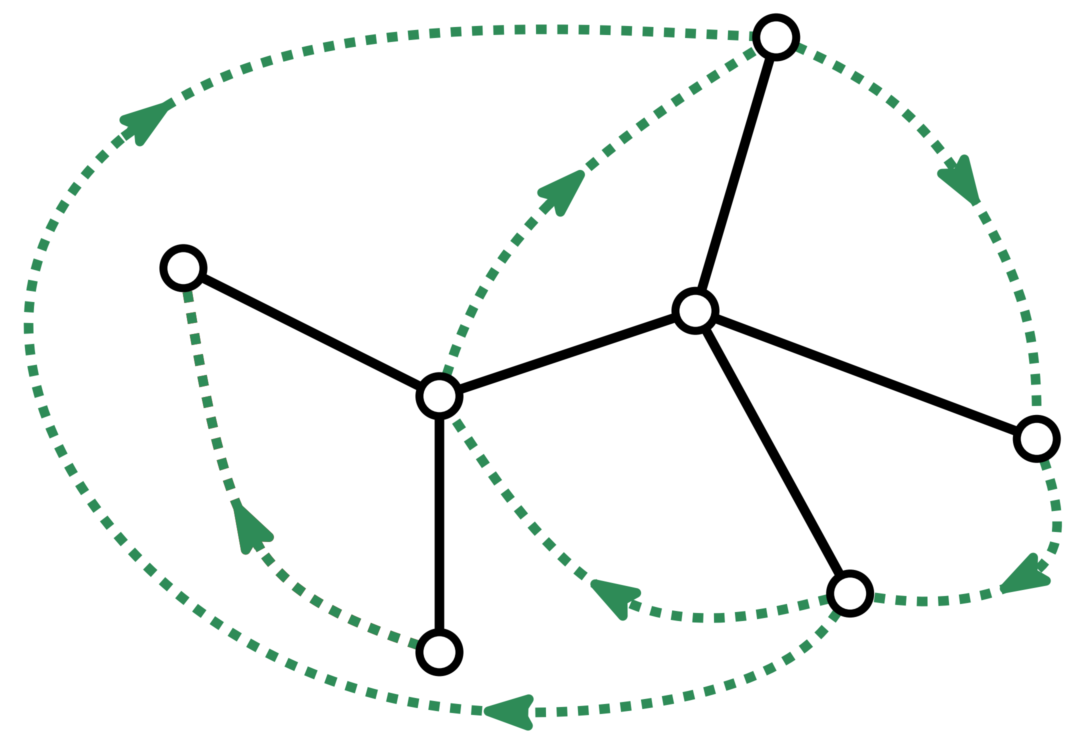

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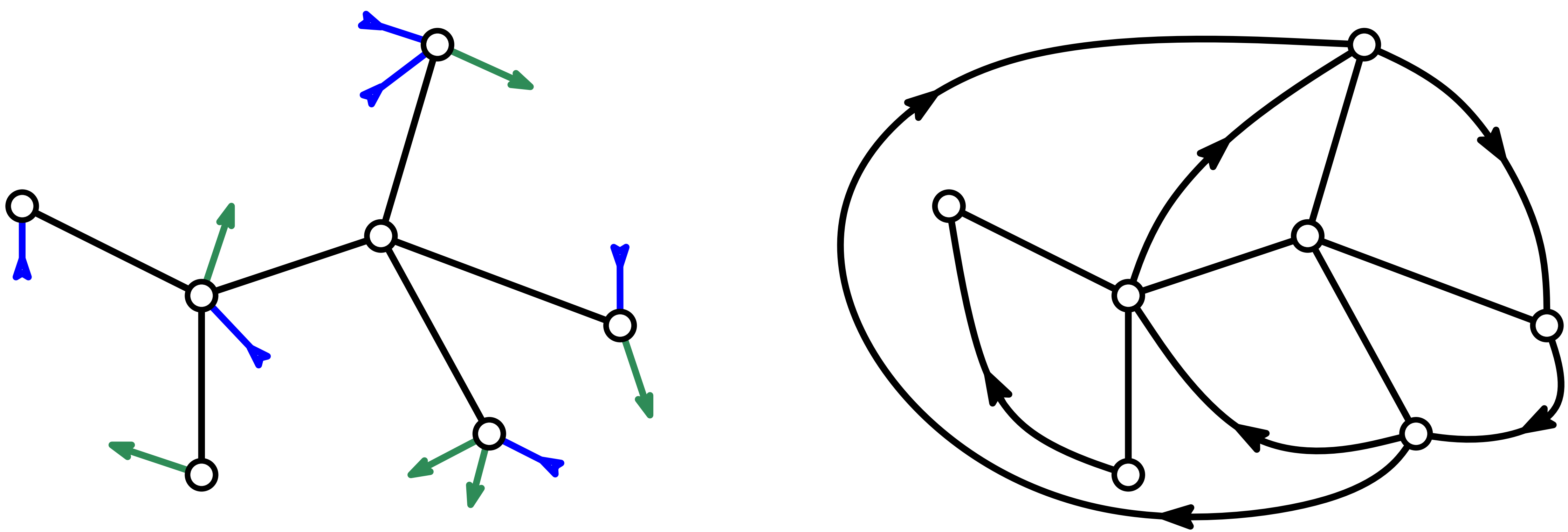
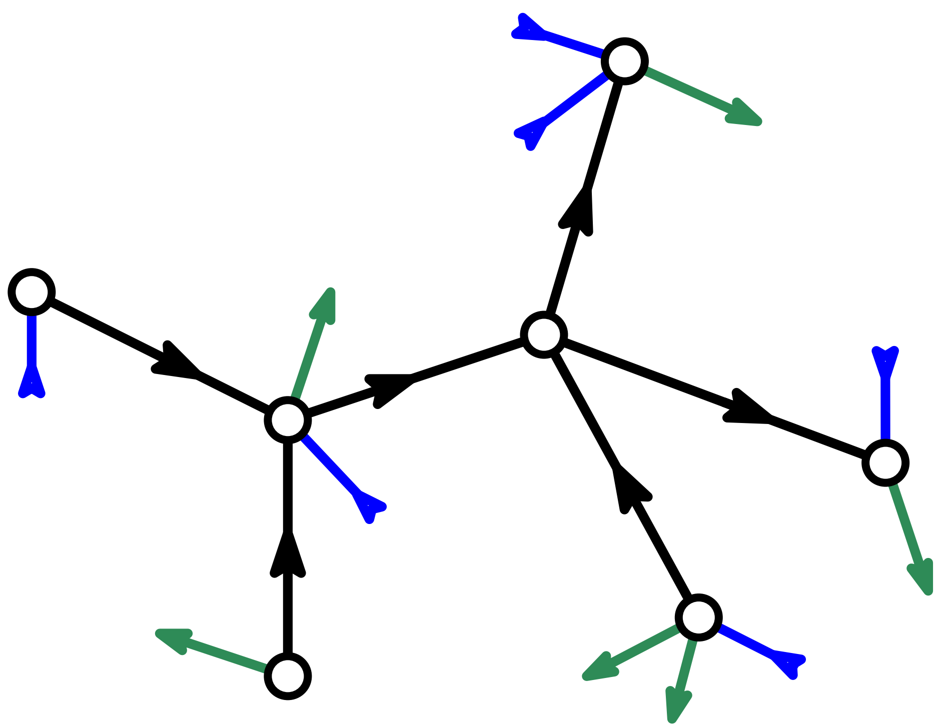
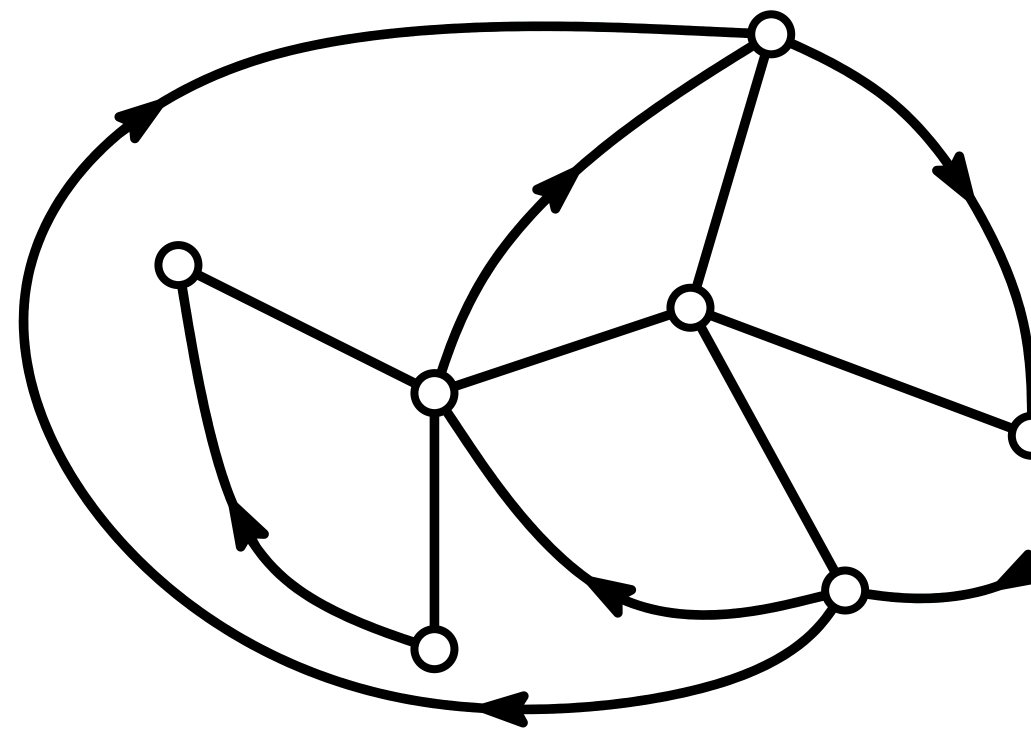


A plane map can be canonically associated to any blossoming tree by making all closures clockwise.

What is a blossoming tree ?





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If the edges of the tree are oriented + closure edges oriented naturally
$\Rightarrow$ **orientation of the map without ccw cycles**.

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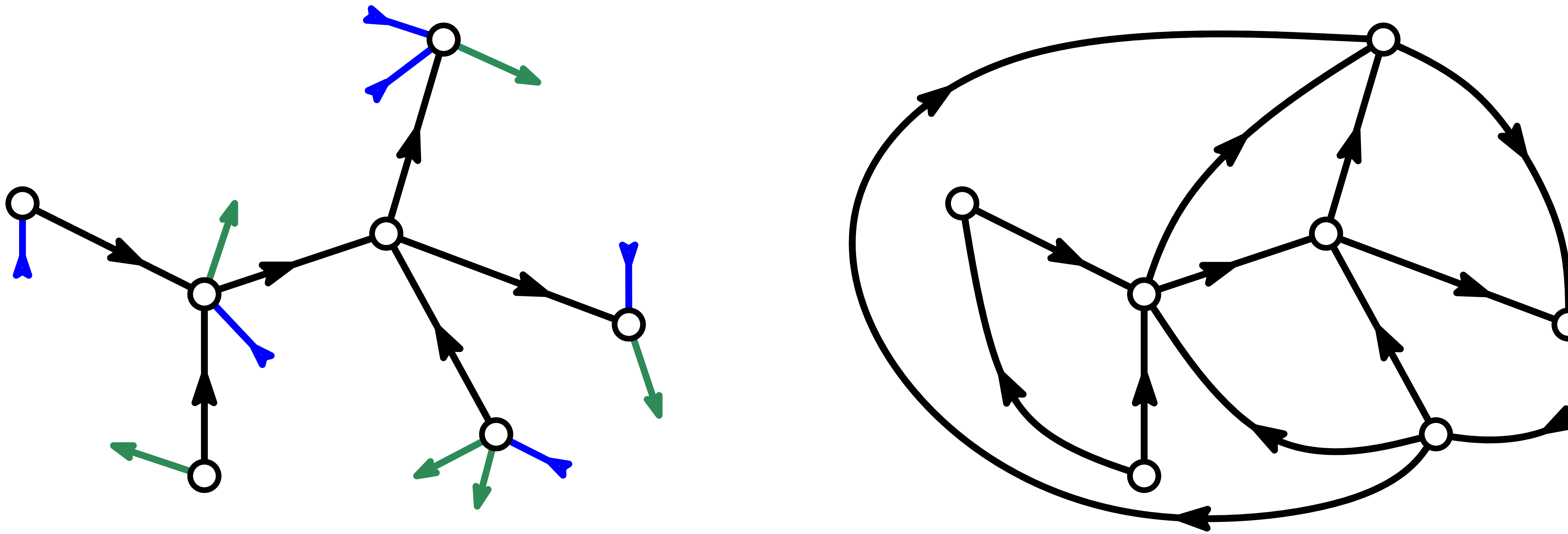


A plane map can be canonically associated to any blossoming tree by making all closures clockwise.

If the edges of the tree are oriented + closure edges oriented naturally
⇒ **orientation of the map without ccw cycles**.

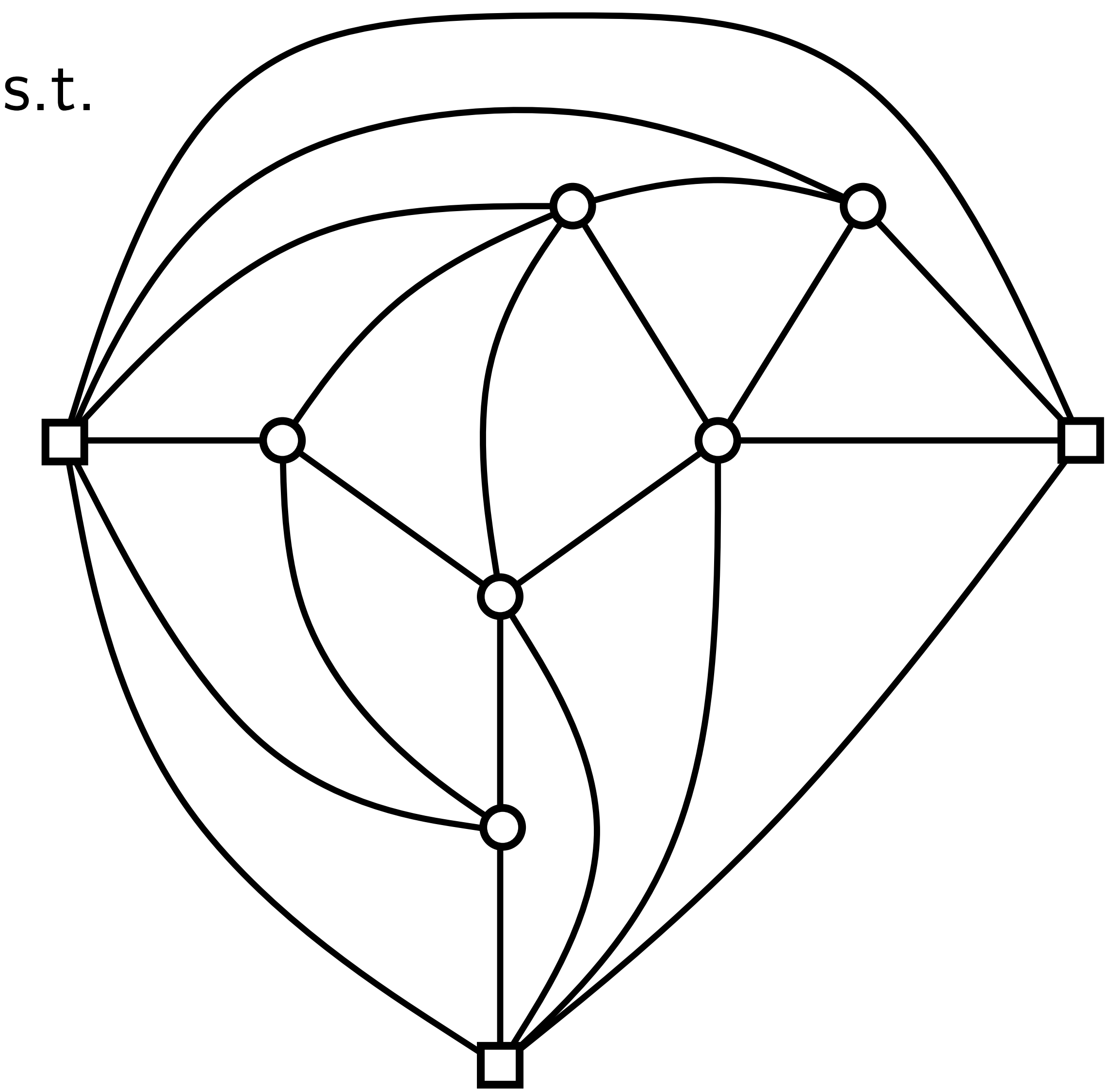
Orientations

First: orientation for **simple triangulations**

3-orientation $=$ orientation of the edges s.t.

out(□) $= 1$

out(○) $= 3$



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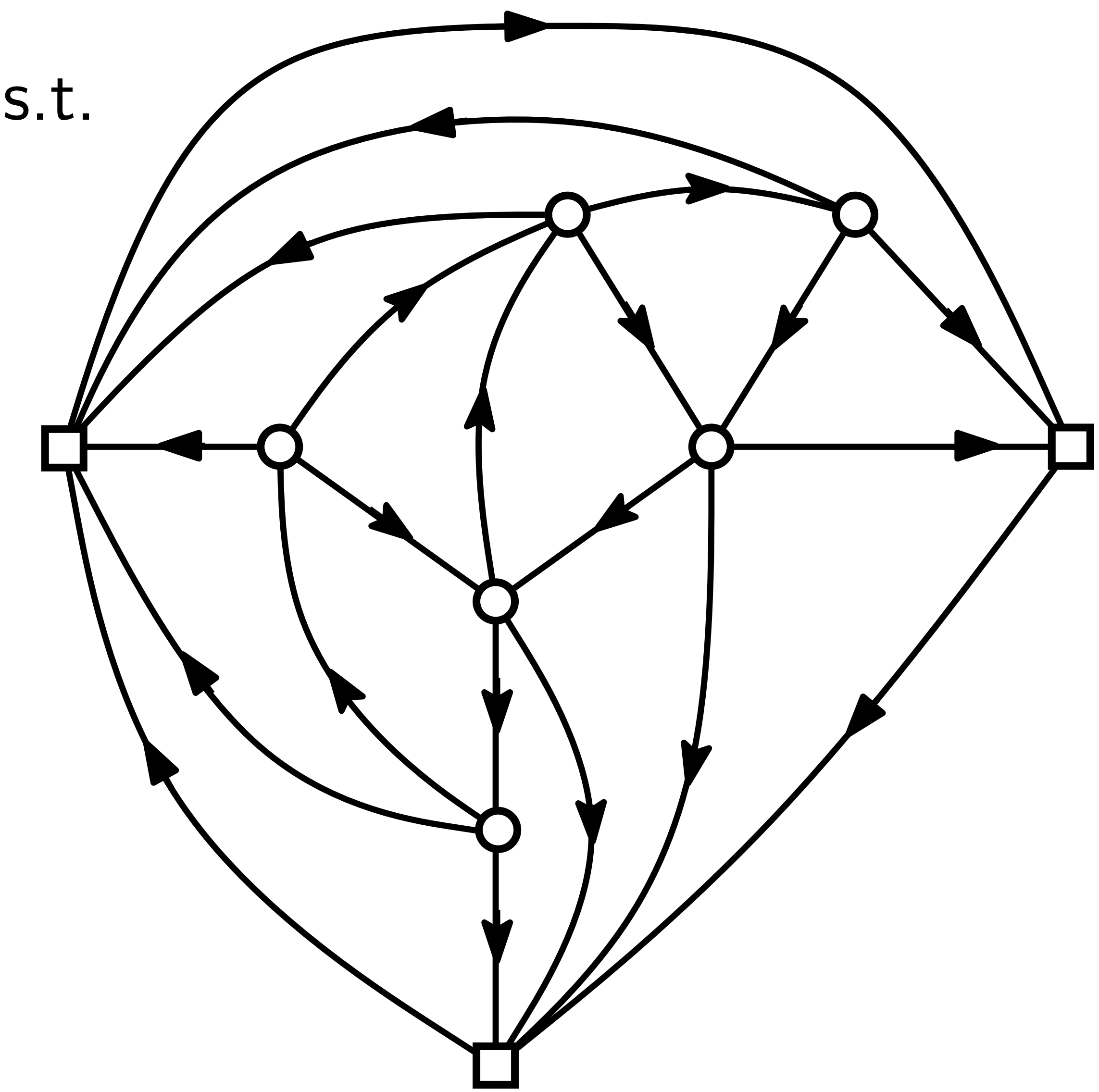

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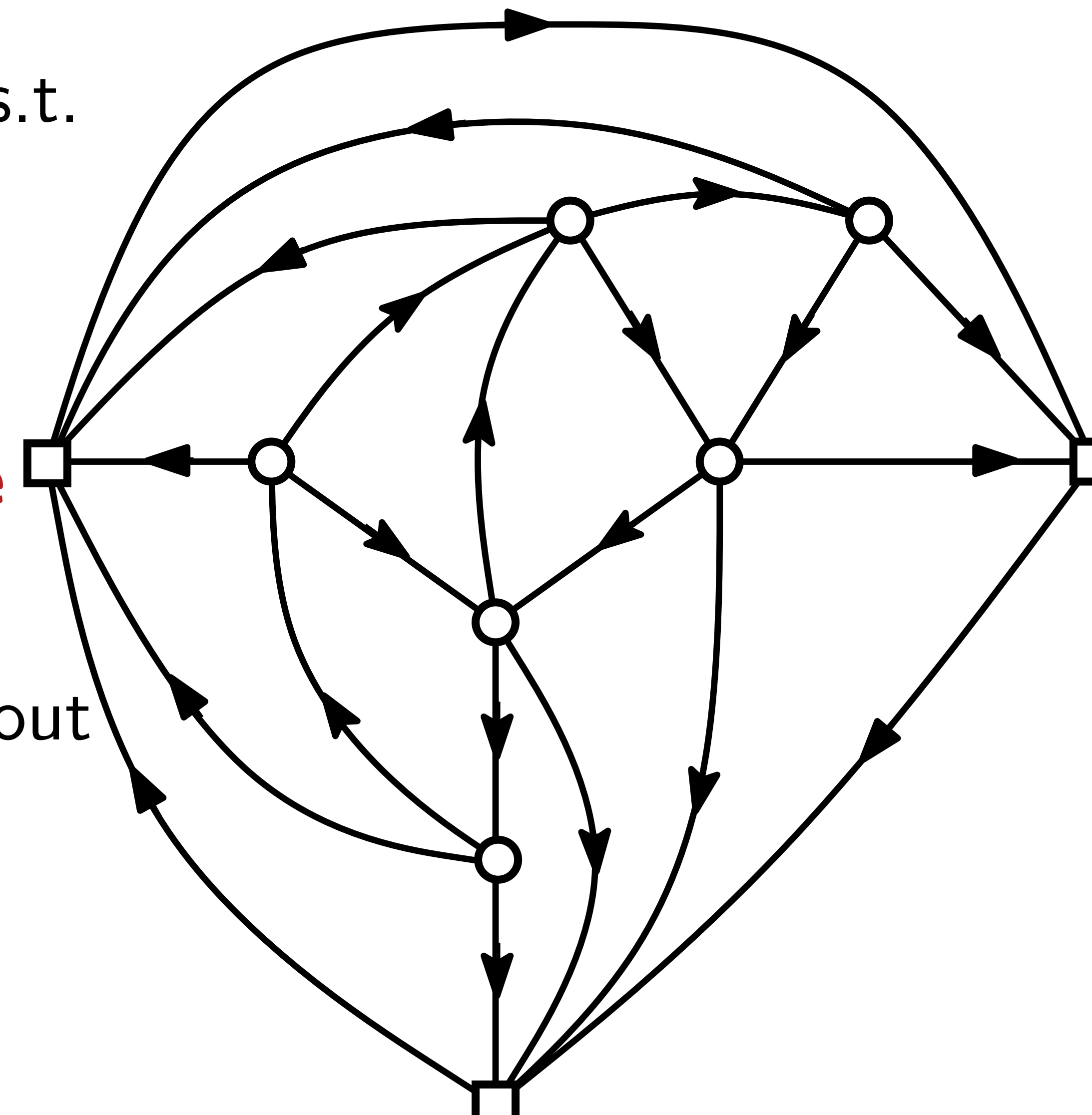
These orientations characterize simple triangulations [Schnyder]

Moreover, there exists a **unique** one without counterclockwise cycles.



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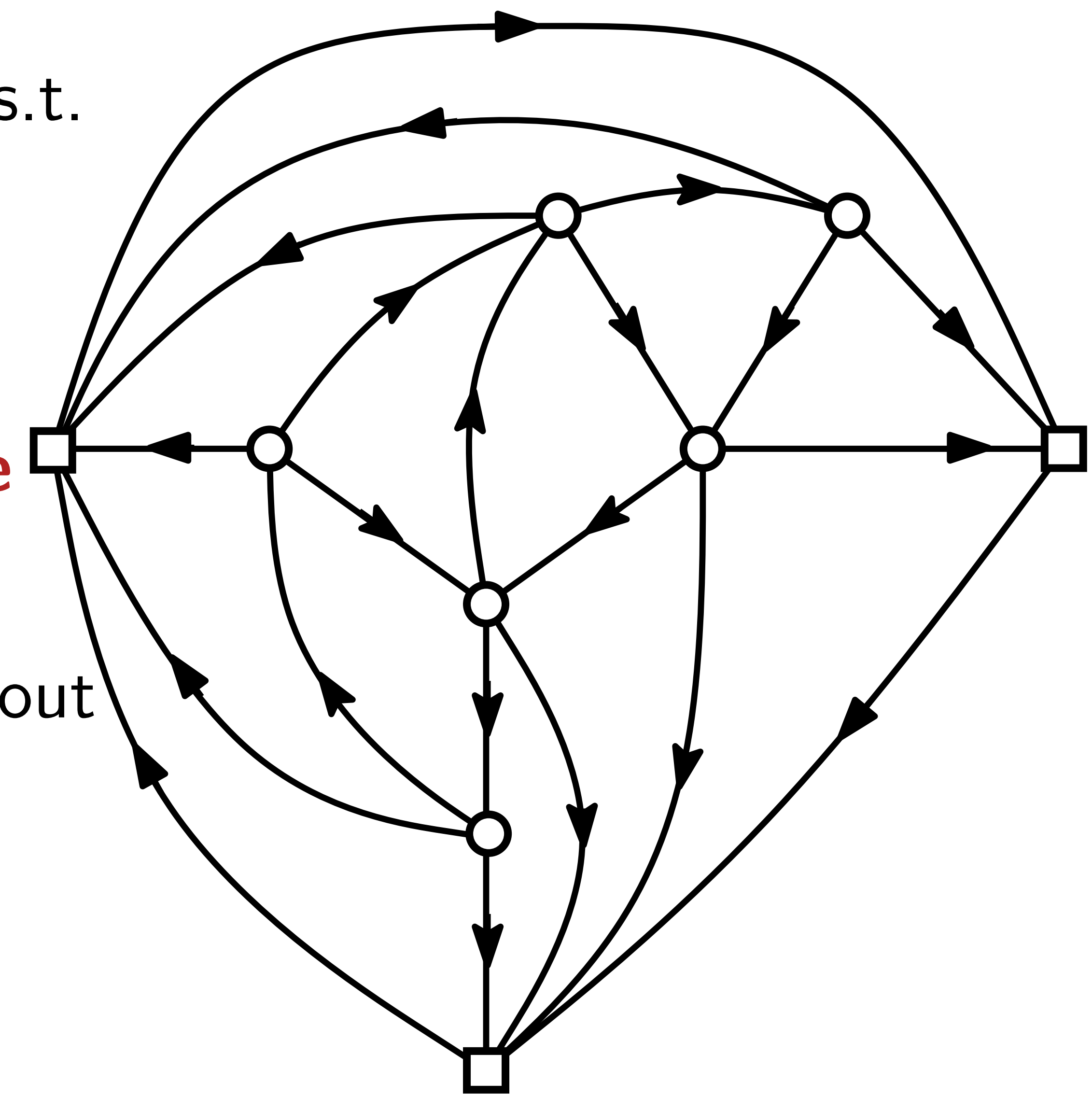
What about **general simple maps ?**



Orientations

Next: orientation for **simple outer-triangular maps**

3-orientation with buds $=$
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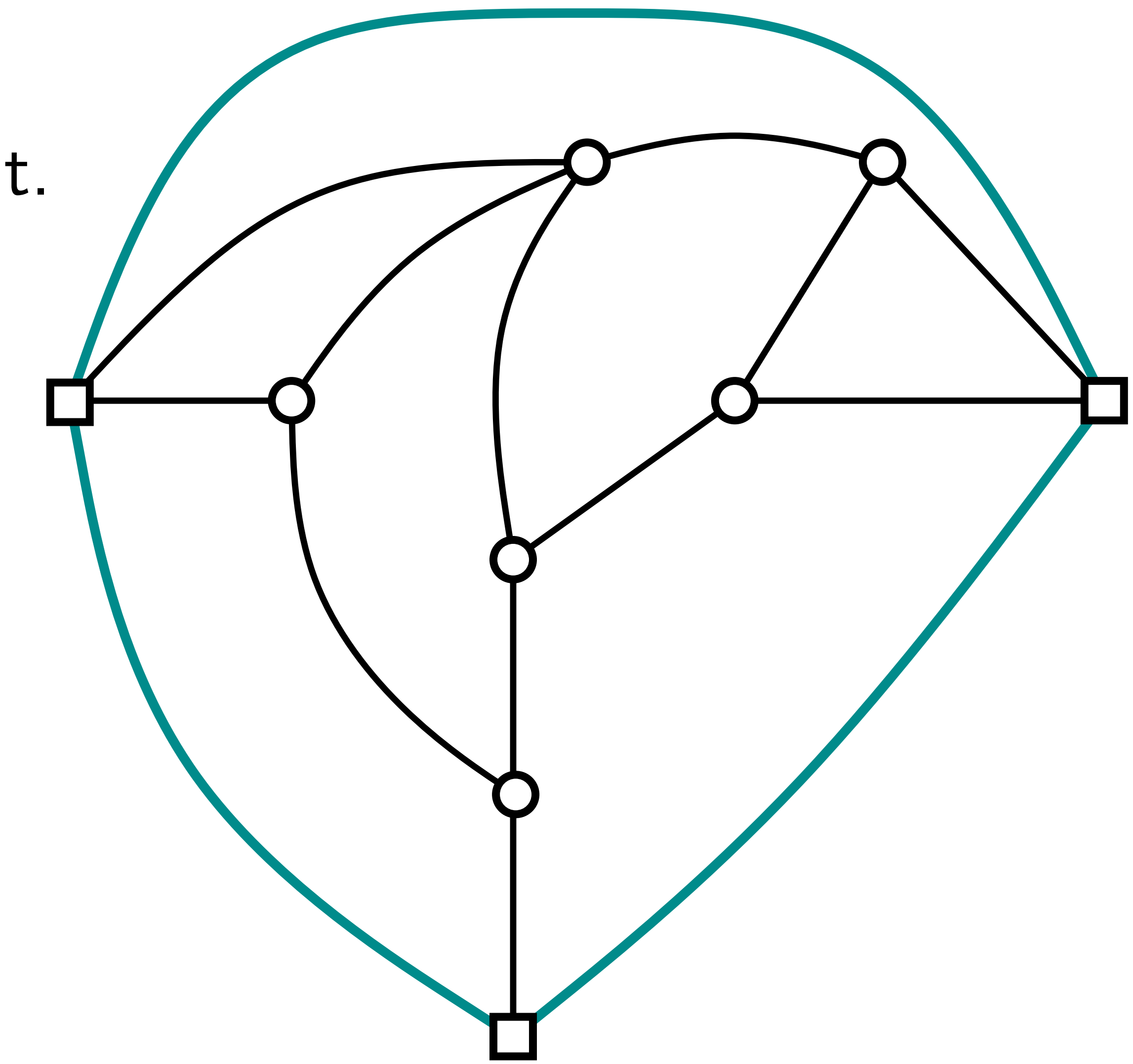

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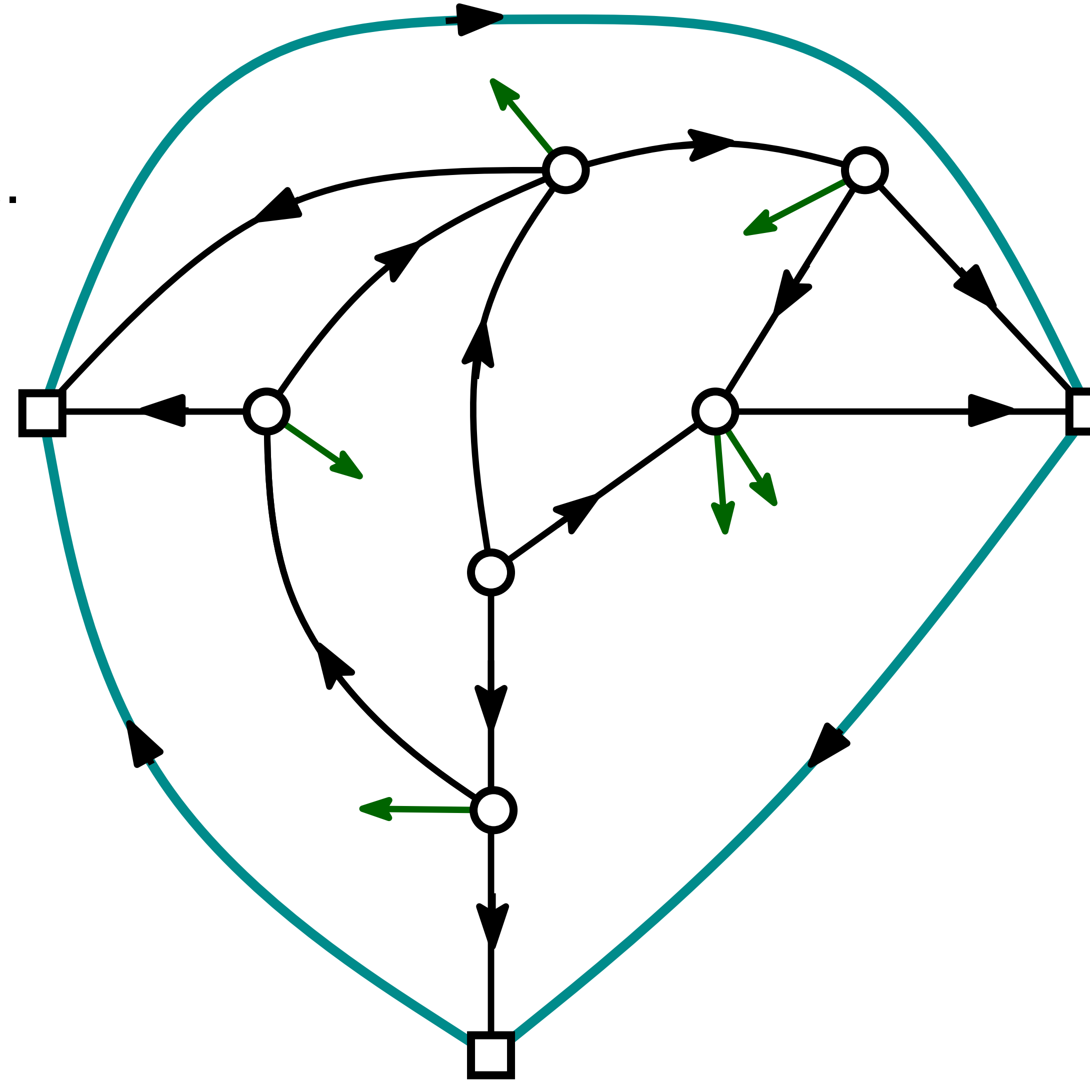

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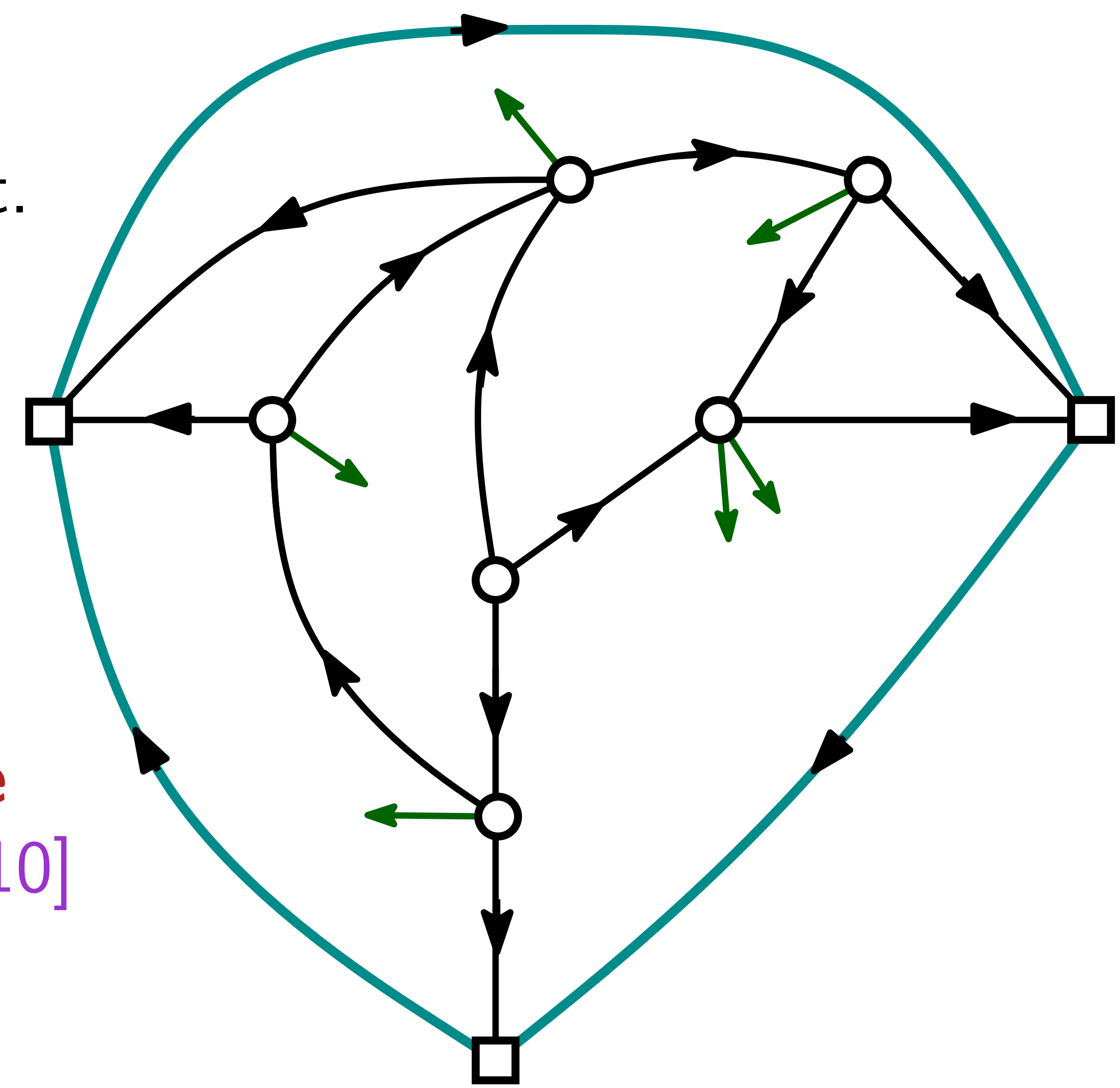
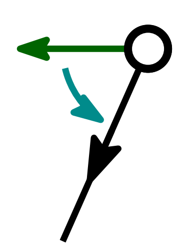


These orientations characterize simple outer-triangular maps [Bernardi, Fusy '10]

Moreover, there exists a **unique** one without counterclockwise cycles and local configuration:



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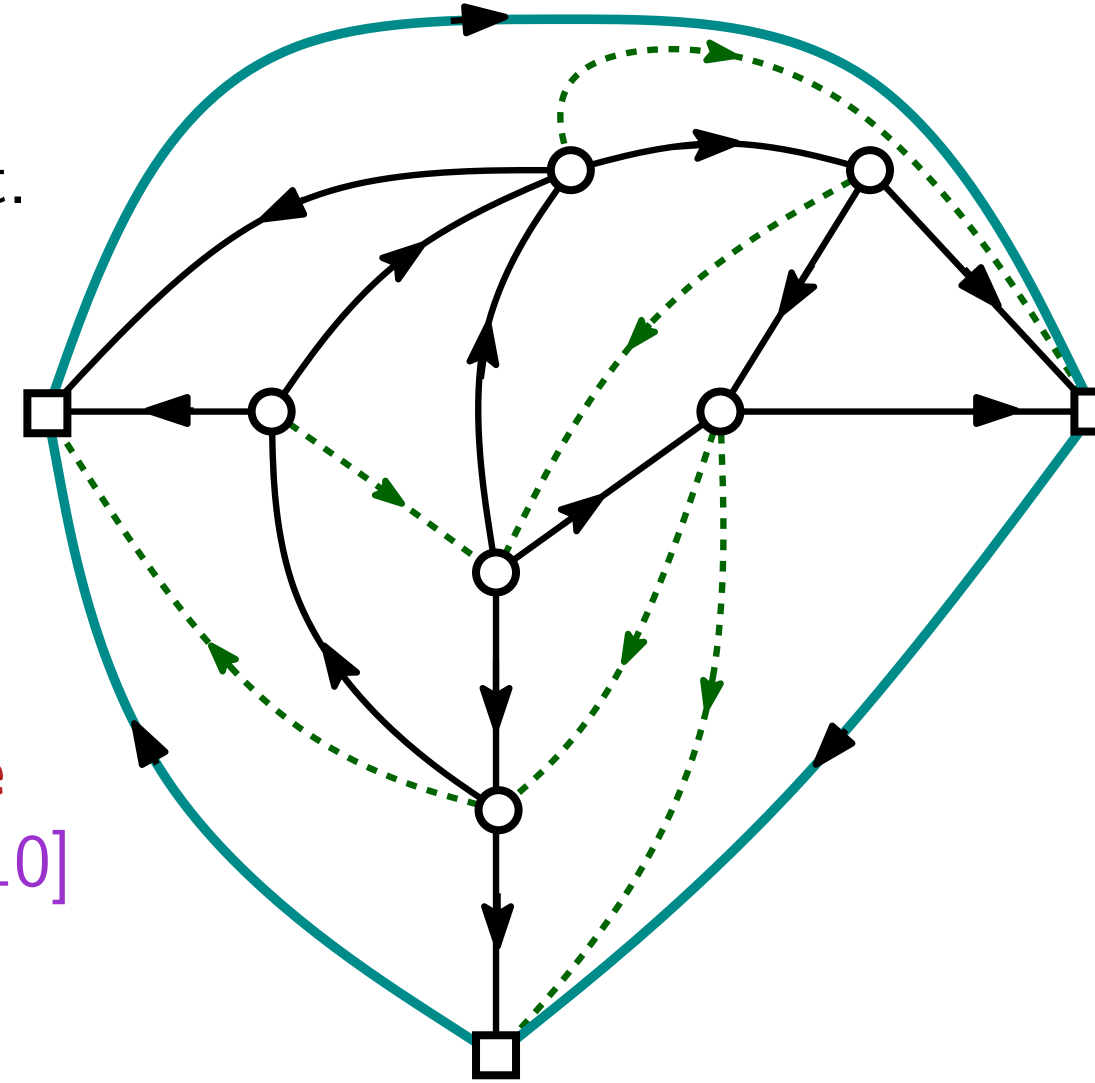

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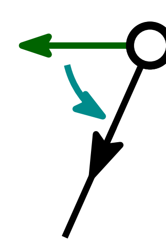
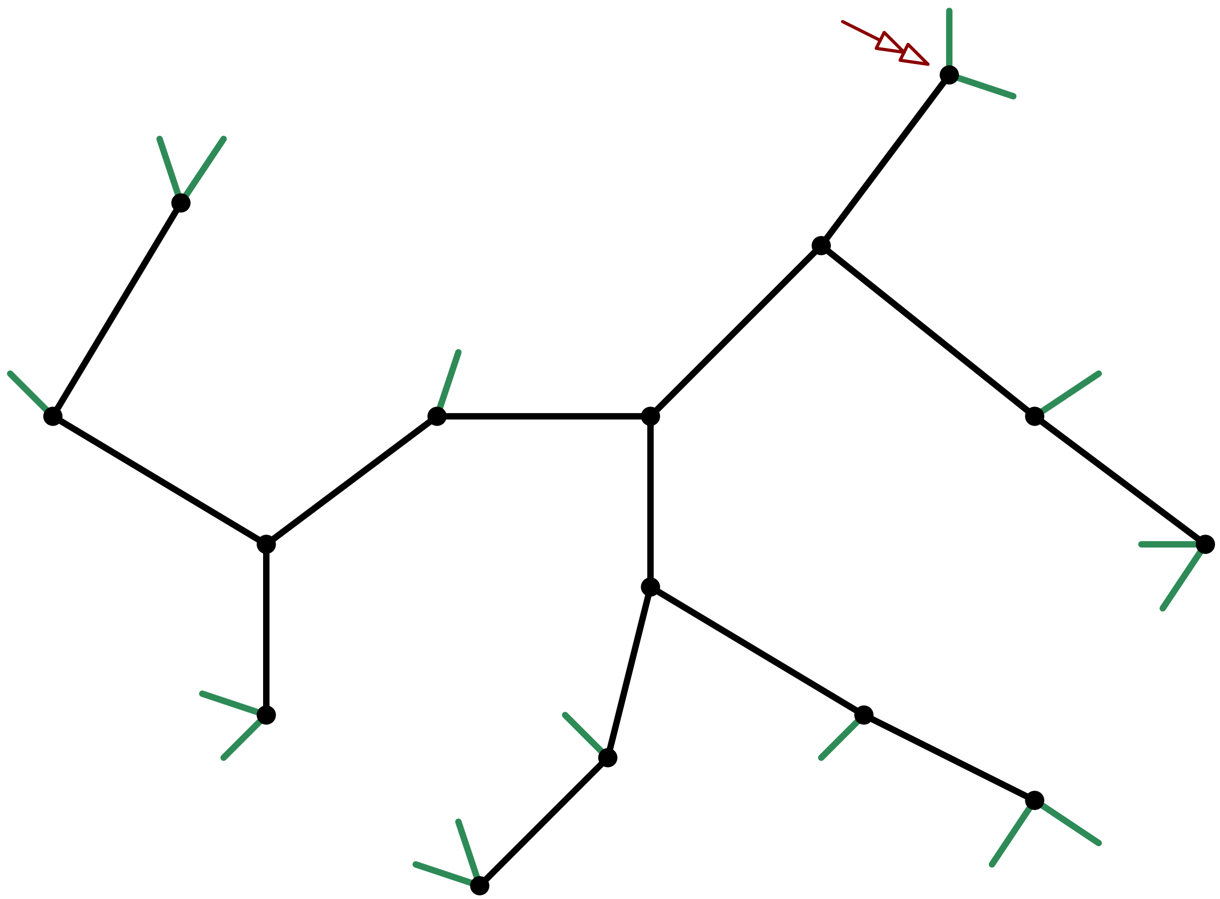


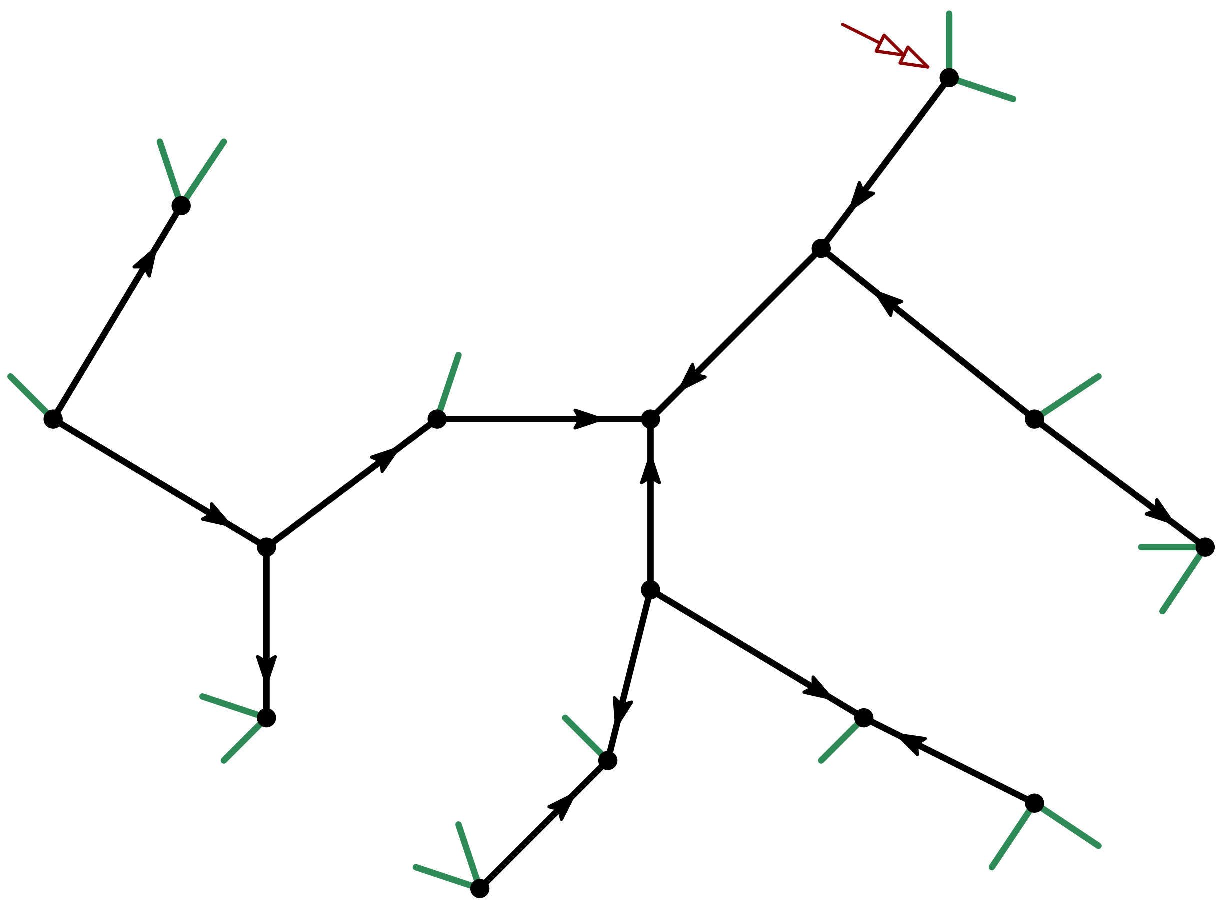
⇒ Give a canonical triangulation of a simple map

Oriented binary trees



- Start from a rooted binary tree (each inner vertex has 3 neighbours)

Oriented binary trees



- Start from a rooted binary tree (each inner vertex has 3 neighbours)
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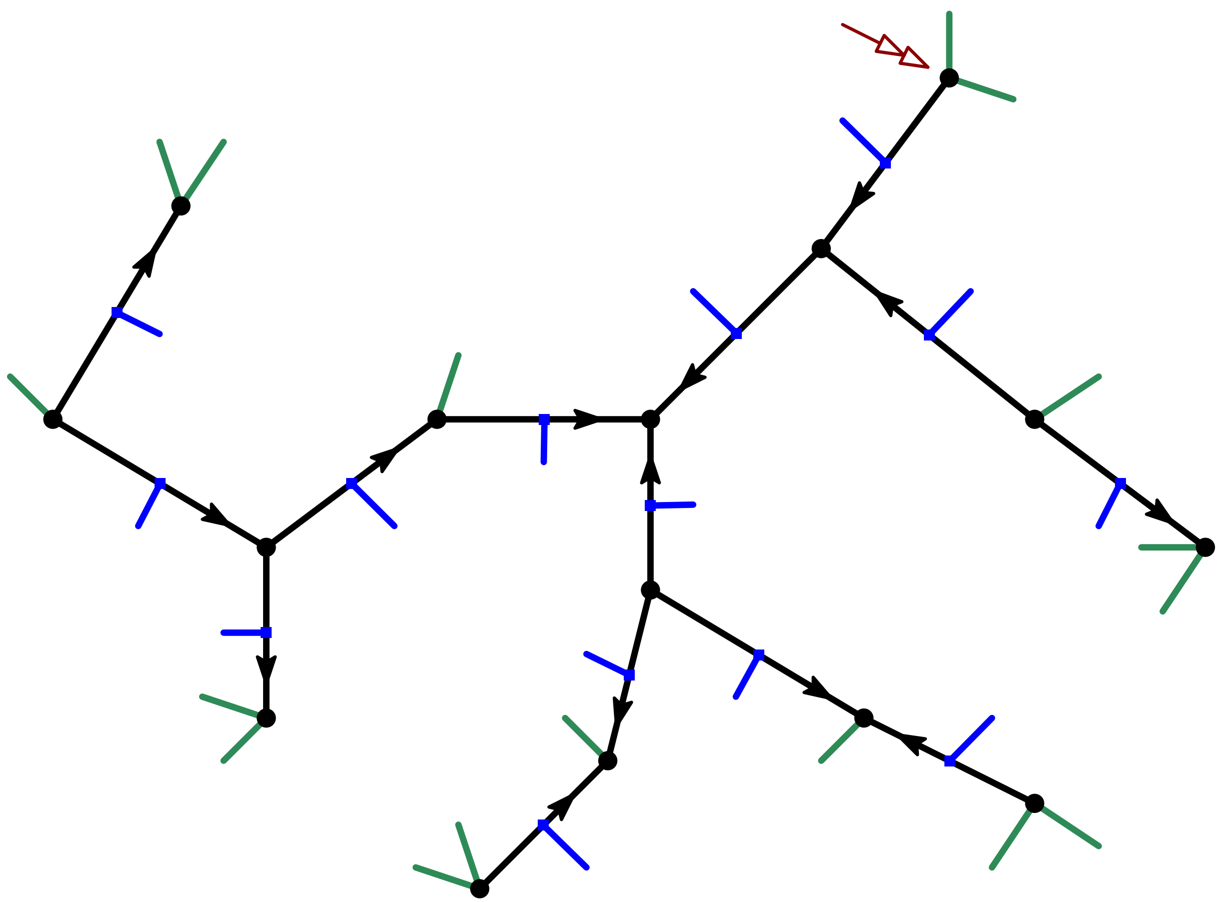

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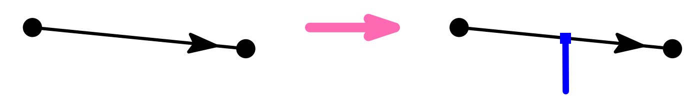

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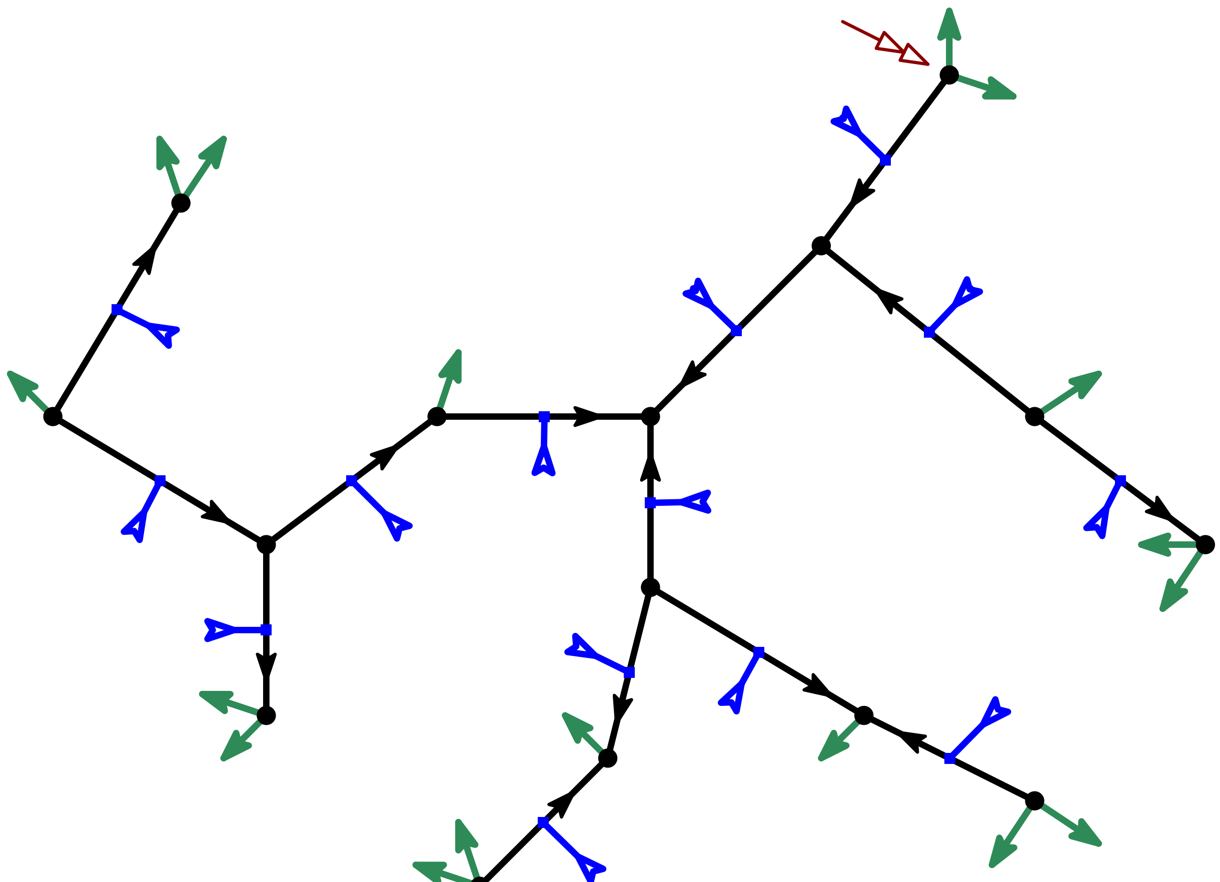

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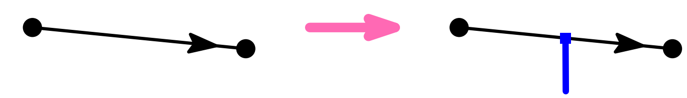
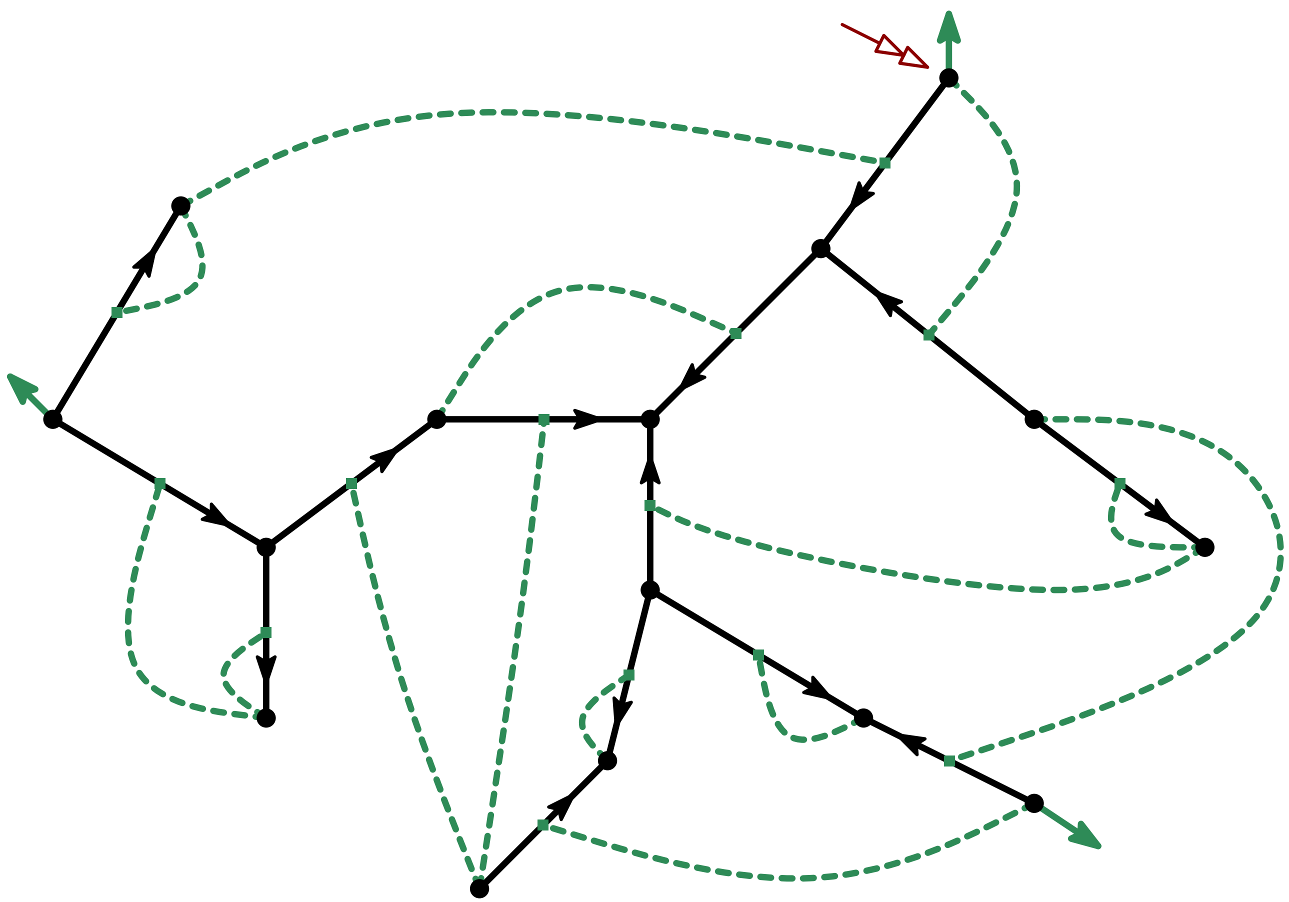


- See — as opening stems and — as closing stems.

Oriented binary trees



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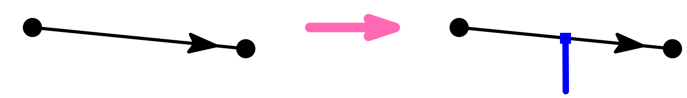
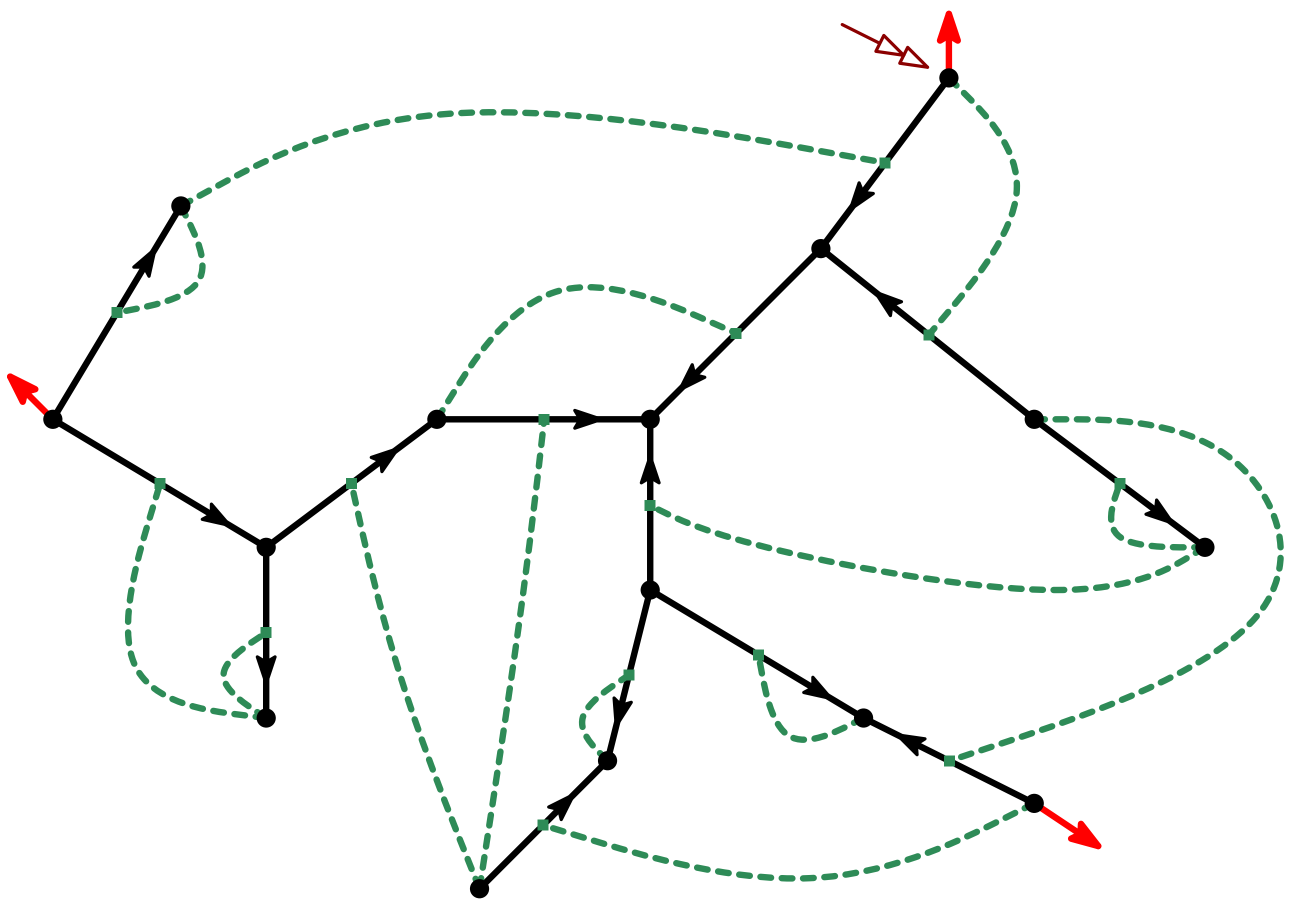


- See as opening stems and as closing stems.
- Make the closure

Oriented binary trees



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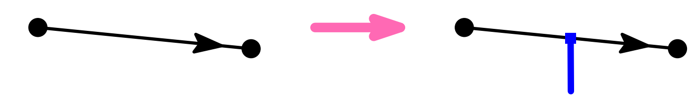
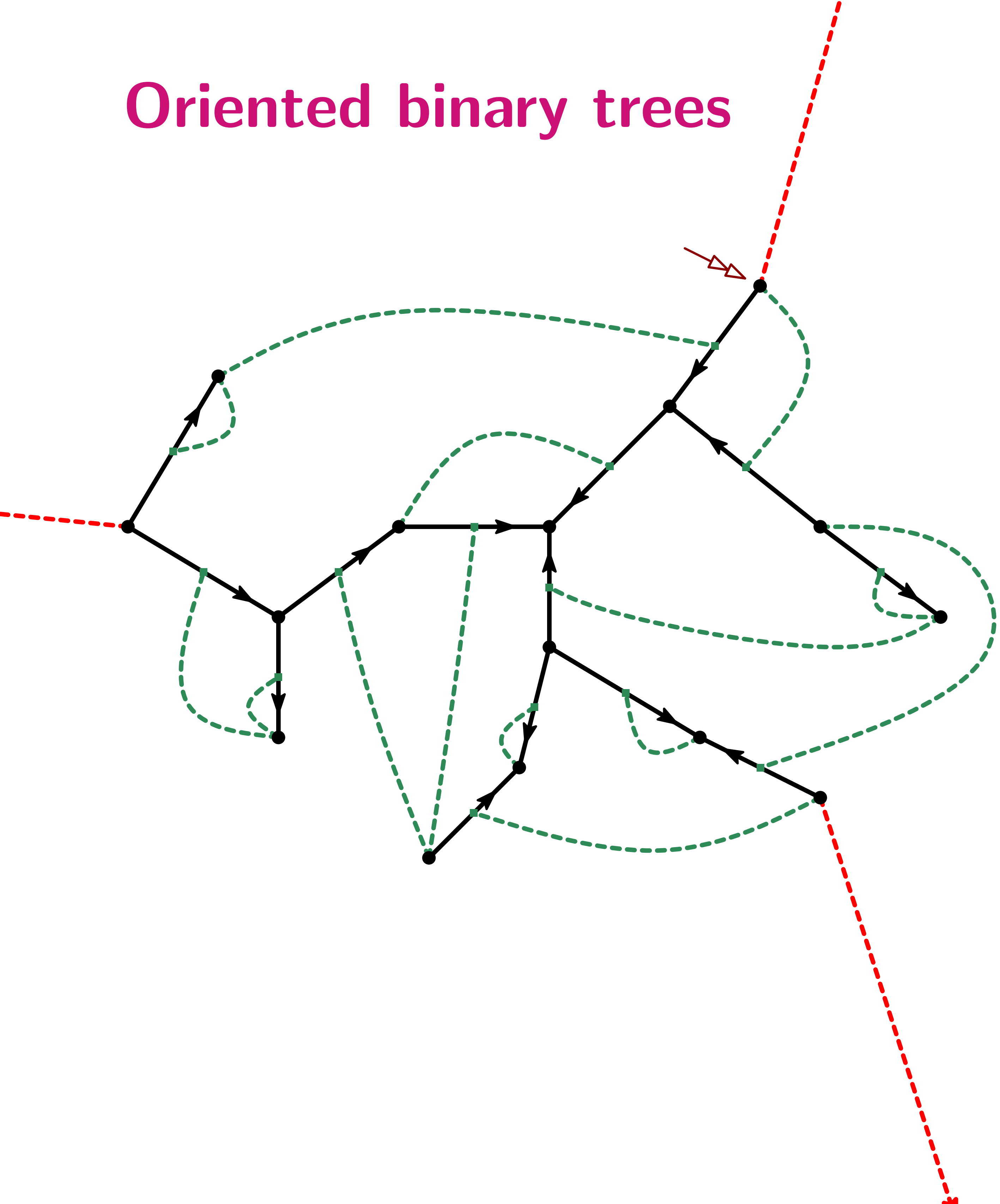


- See as opening stems and as closing stems.
- Make the closure
- 3 opening stems are left unmatched,
- one of them is incident to the root corner: the tree is **balanced**

Oriented binary trees



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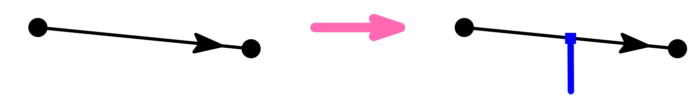
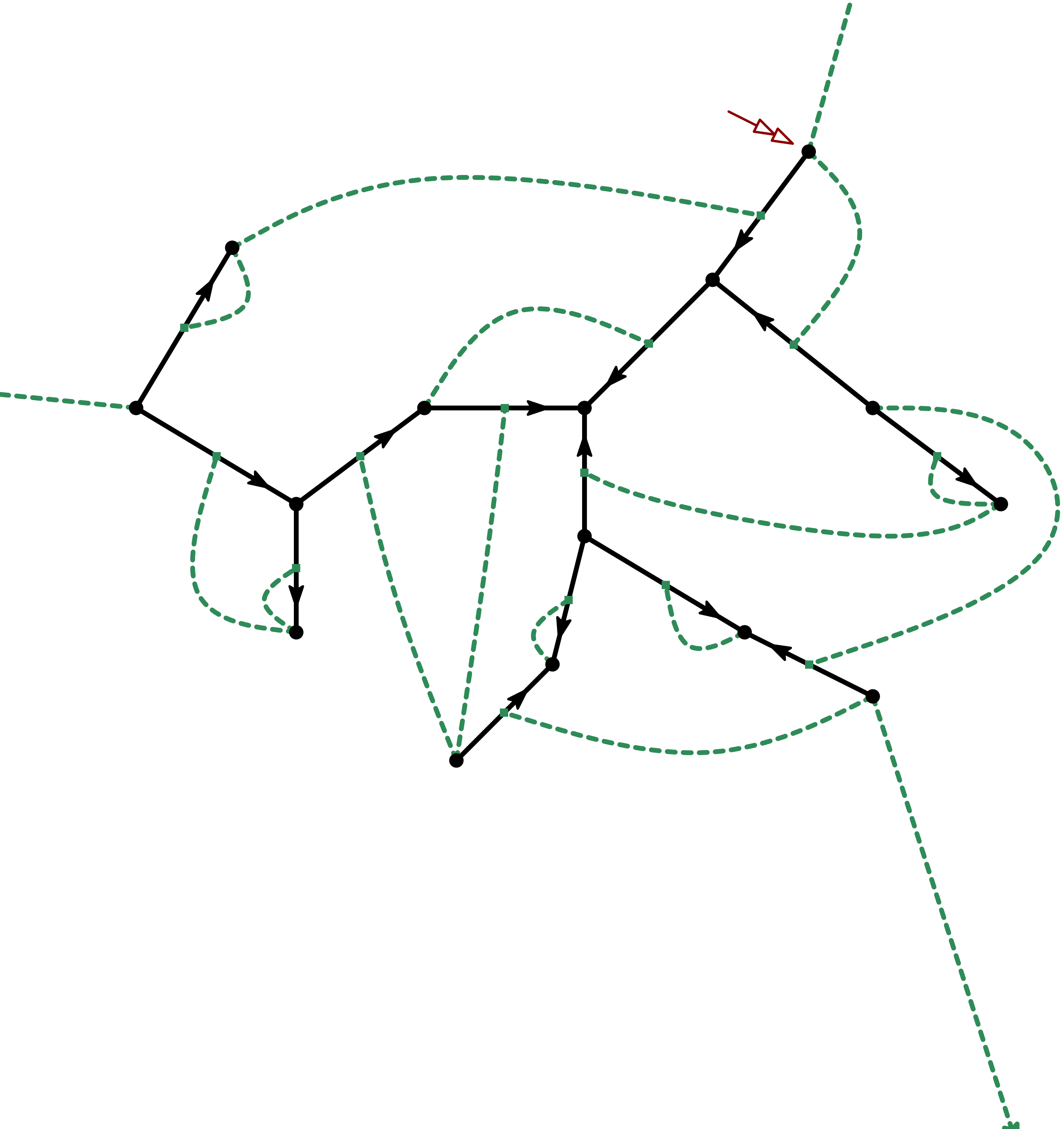


- See as opening stems and as closing stems.
- Make the closure
- 3 opening stems are left unmatched, we close them at ∞
- one of them is incident to the root corner: the tree is **balanced**

Oriented binary trees



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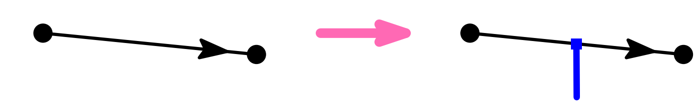
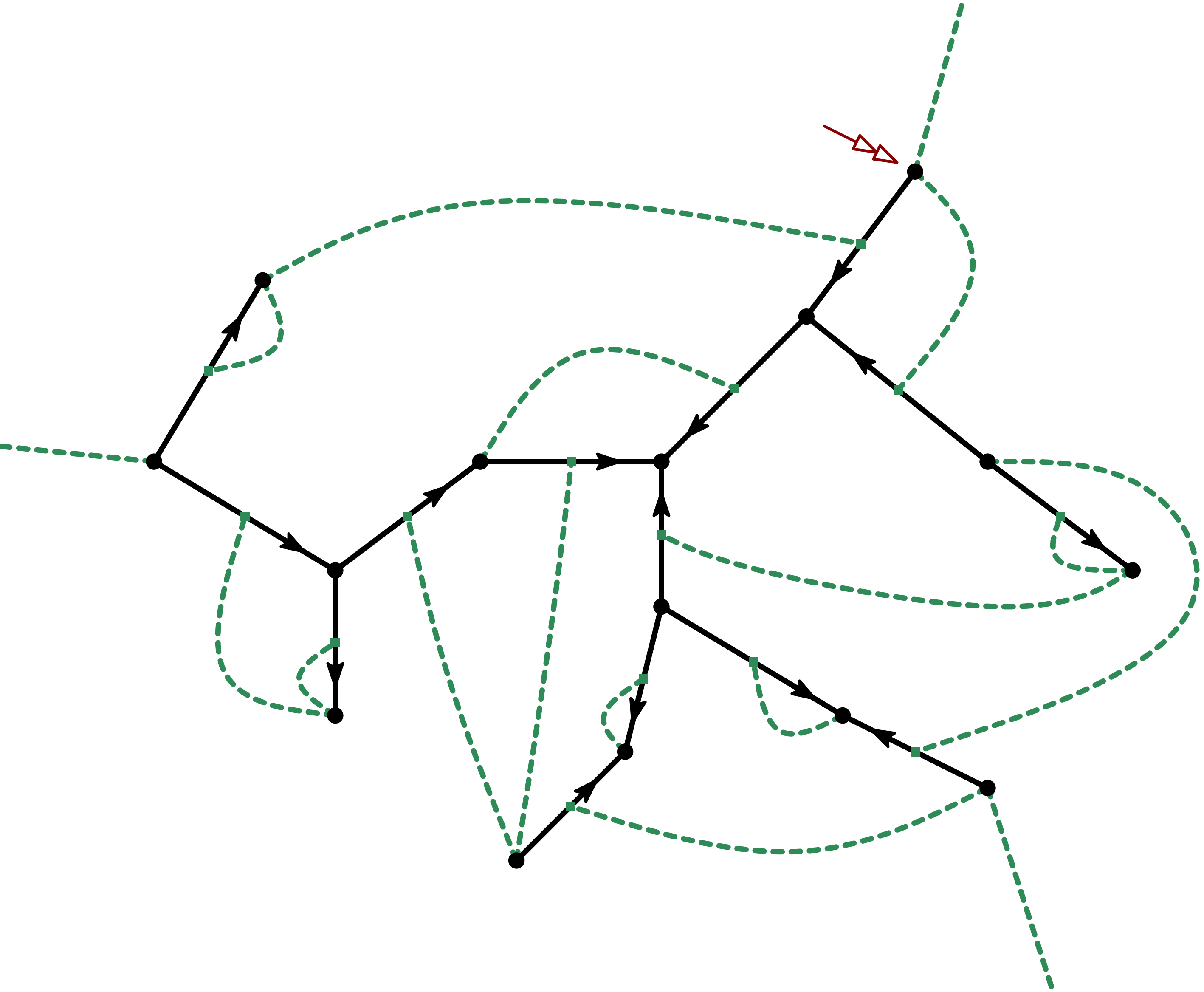


- See —— as opening stems and —— as closing stems.
- Make the closure
- 3 opening stems are left unmatched, we close them at ∞
- one of them is incident to the root corner: the tree is **balanced**
  - We obtain a bipartite cubic map

Oriented binary trees



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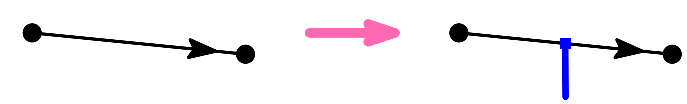

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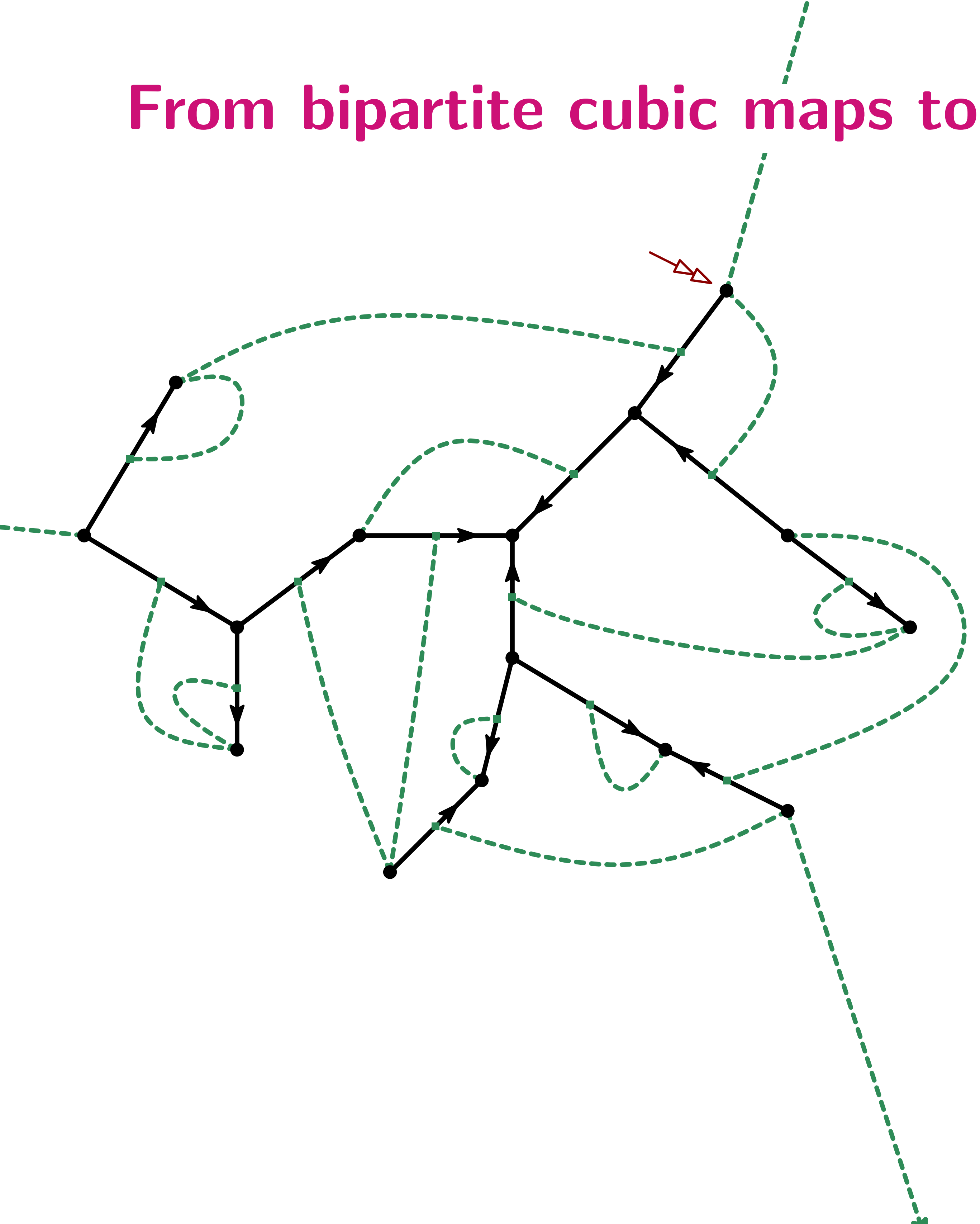
Theorem : [Bousquet-Mélou, Schaeffer '00] This is a bijection between **balanced oriented binary trees** and **bipartite cubic maps**

From bipartite cubic maps to simple maps



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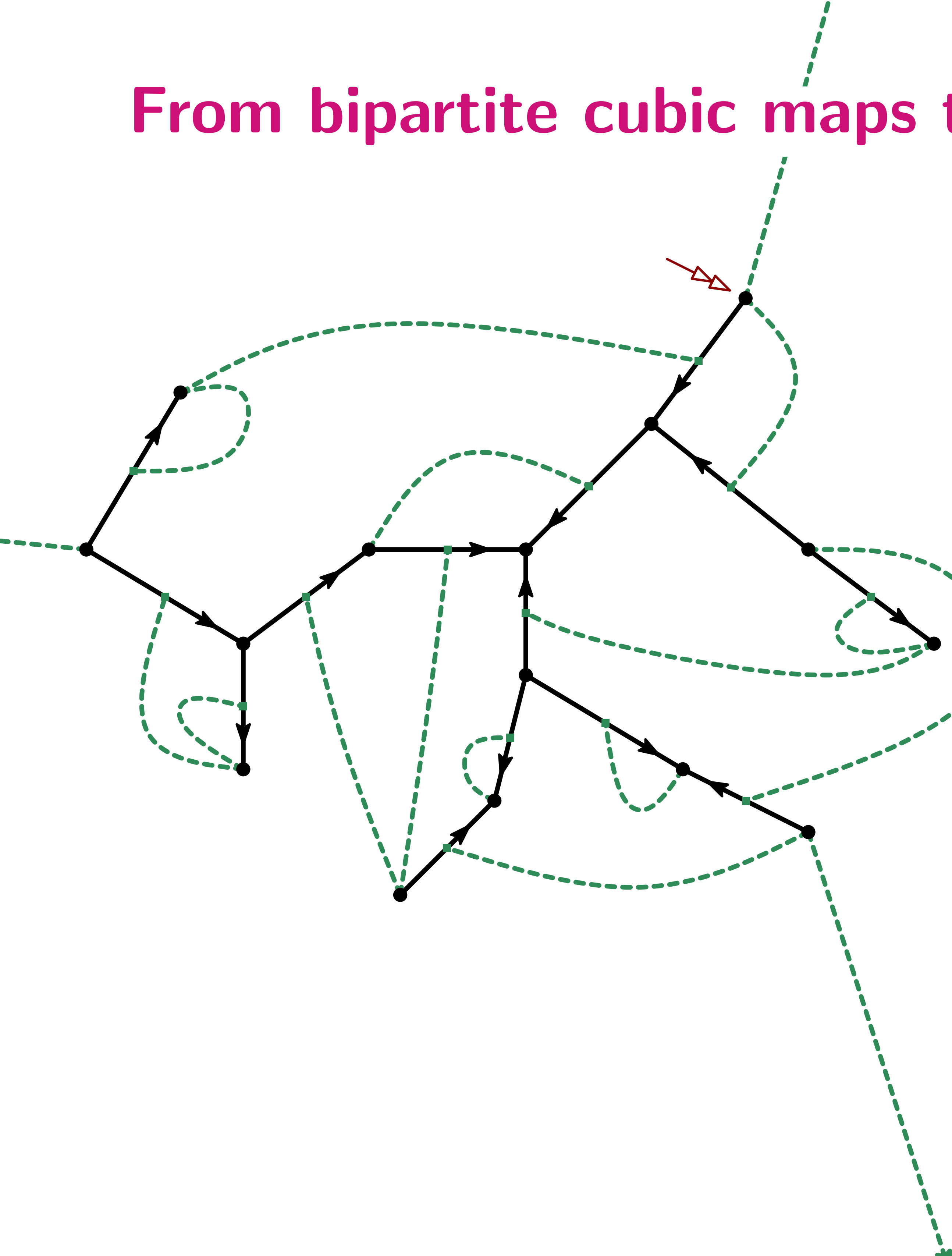

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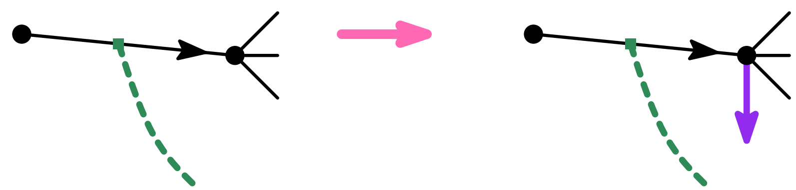

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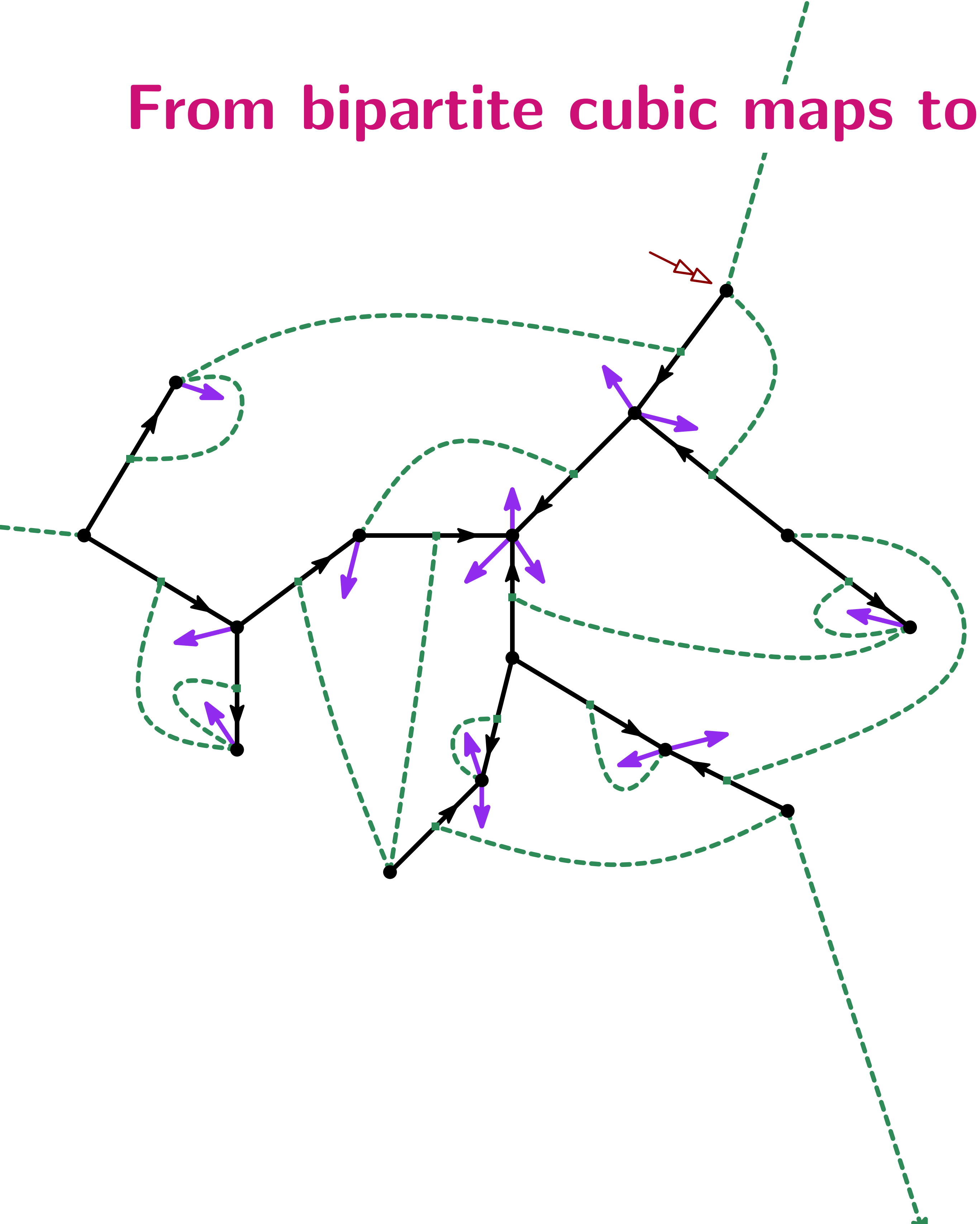

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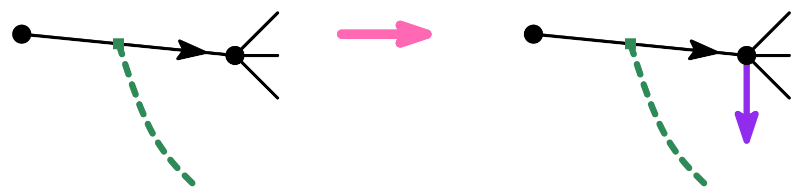

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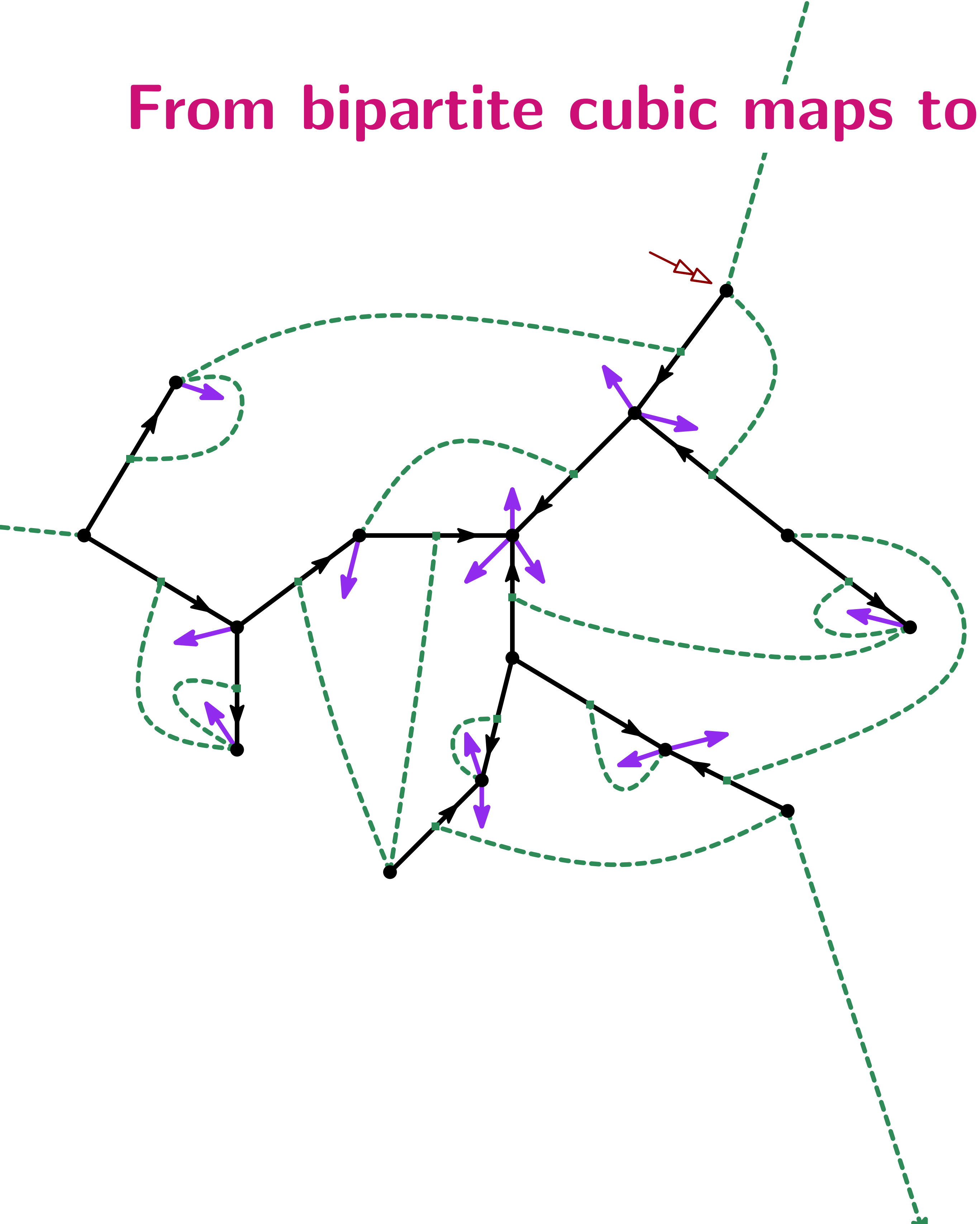

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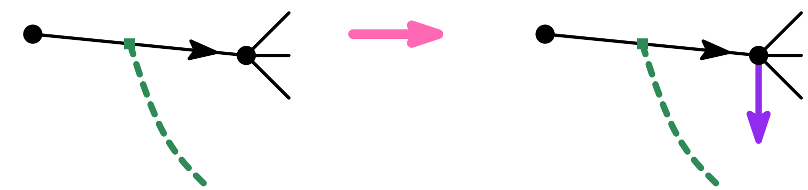
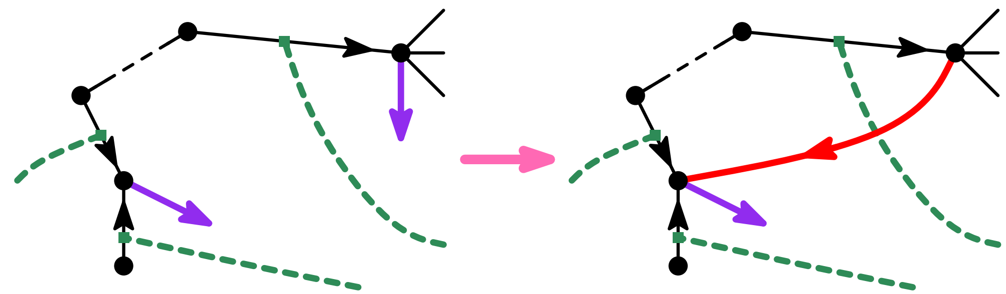


- Turning clockwise around the tree, do the following closures:



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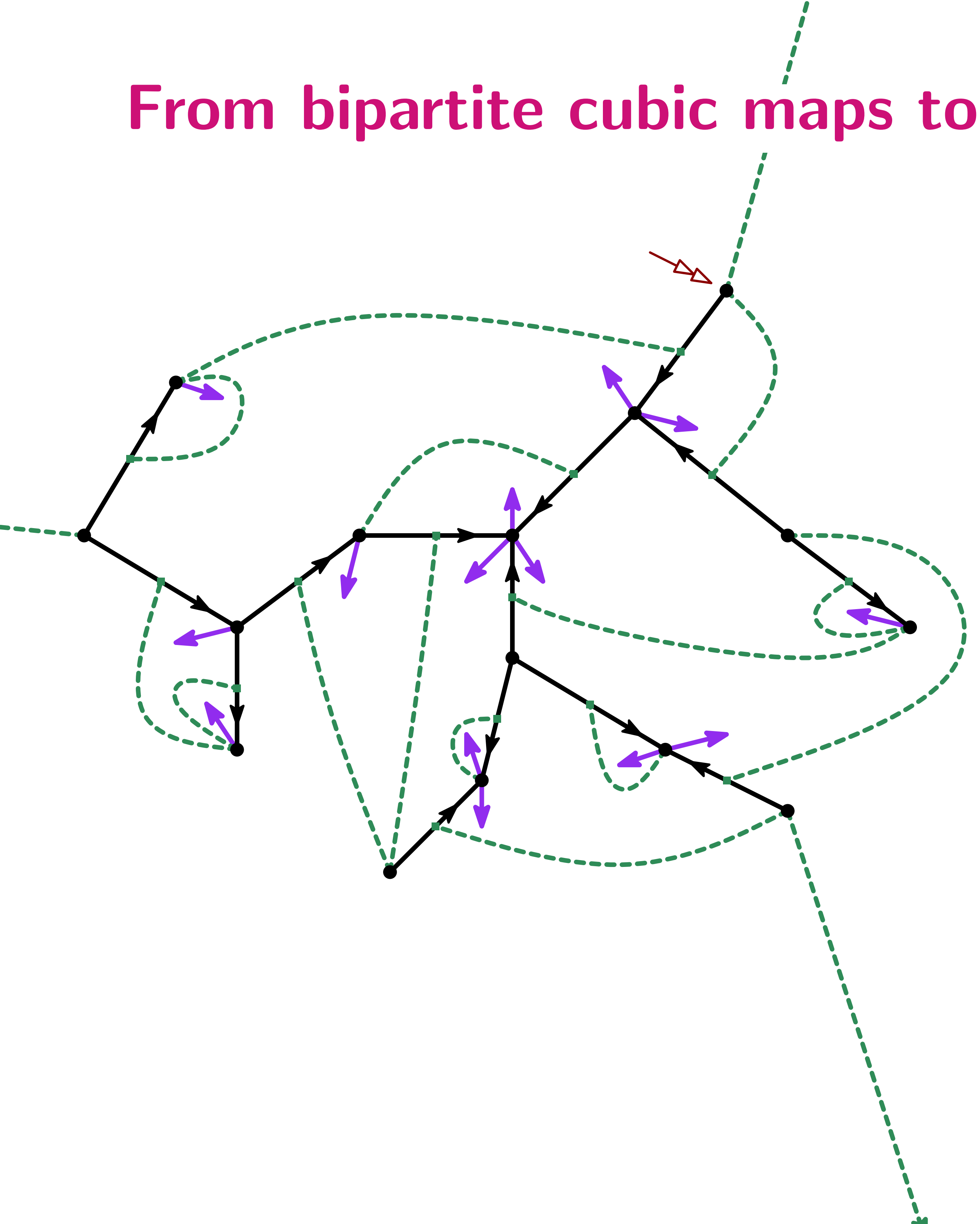

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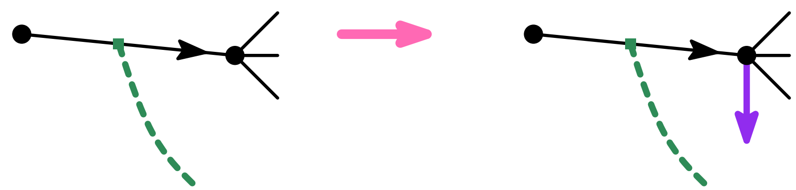

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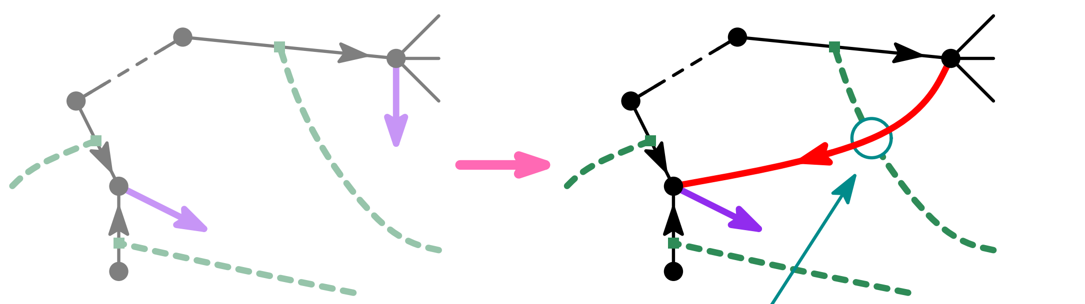

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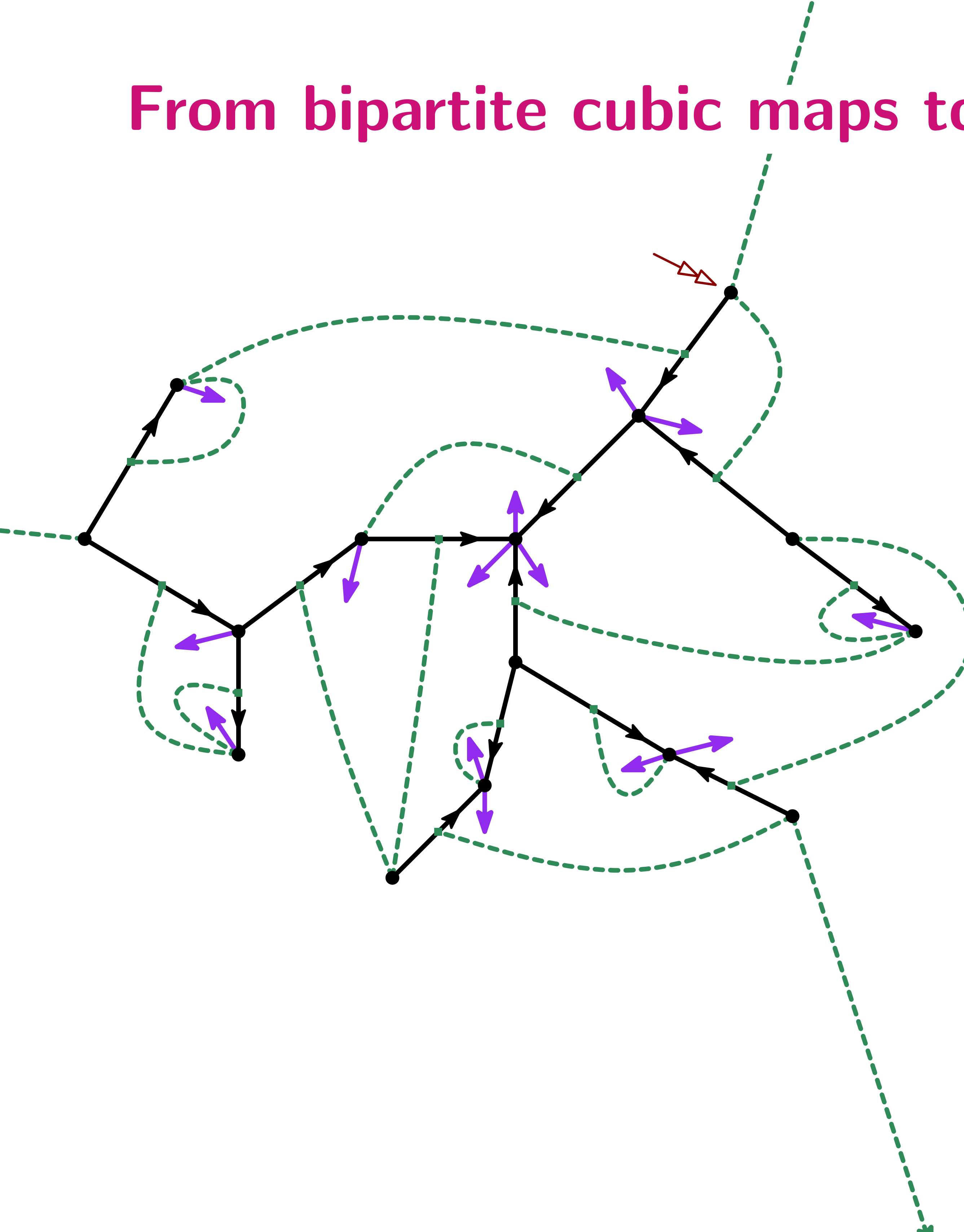

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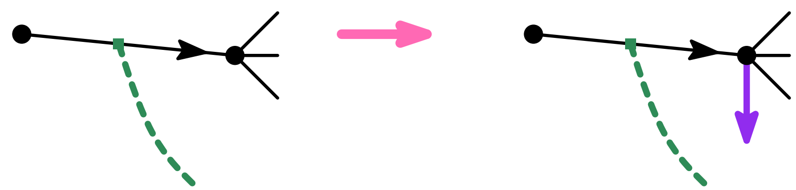

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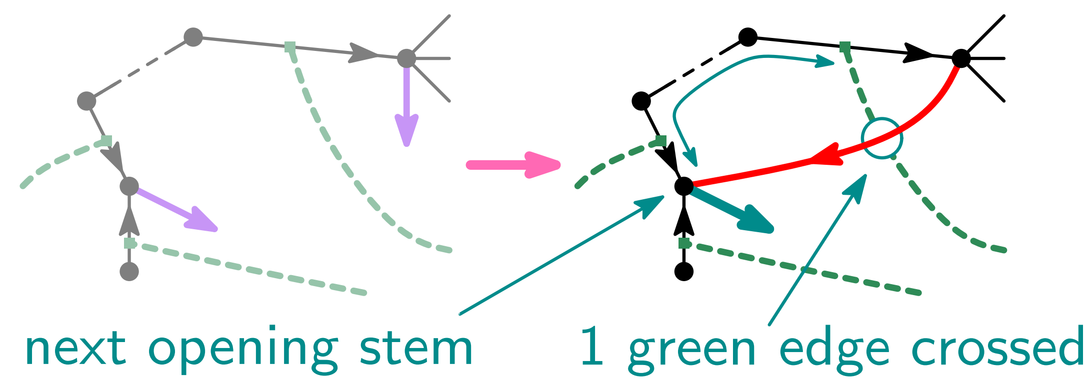

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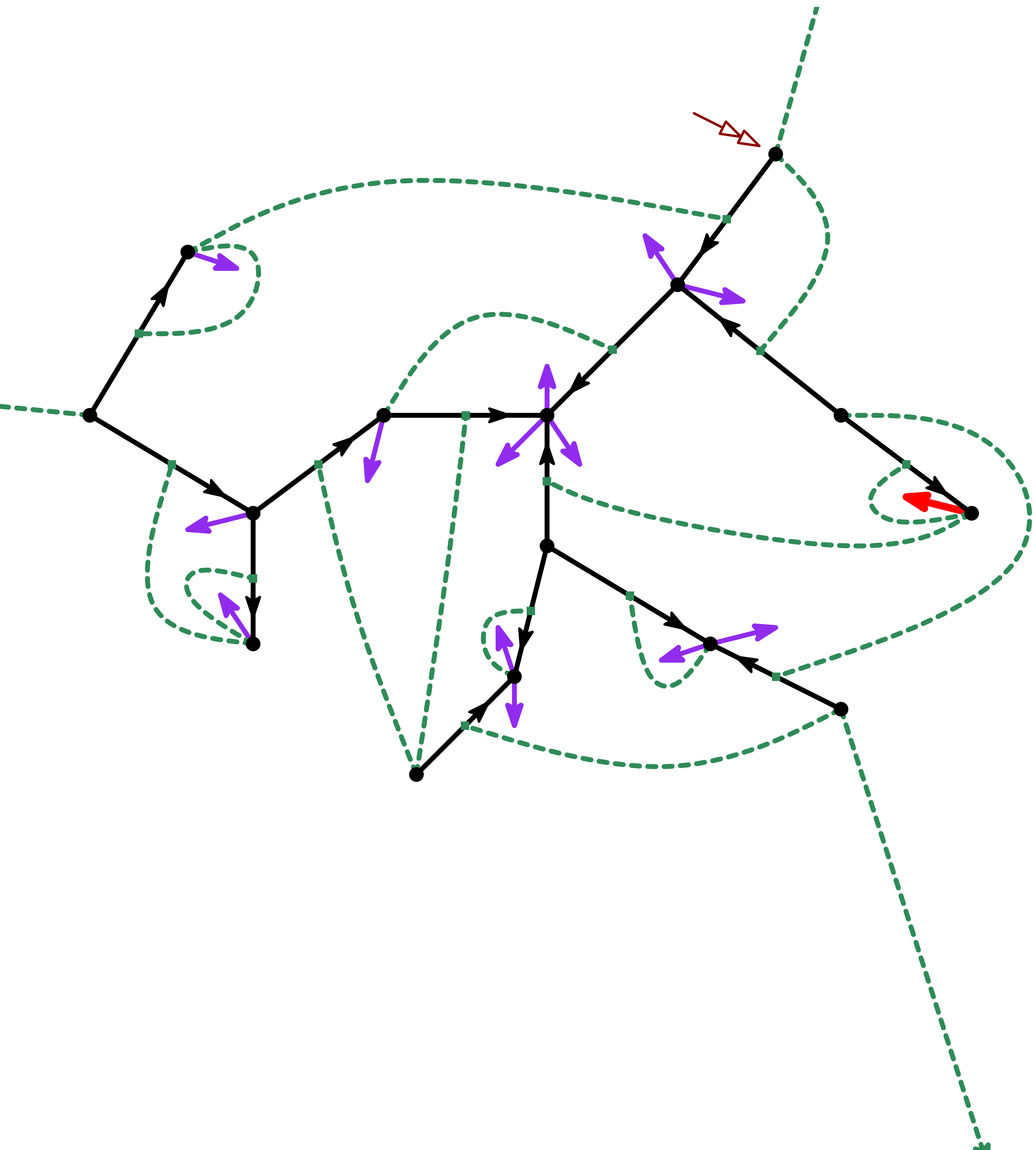

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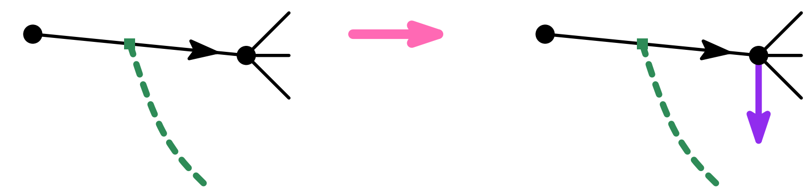

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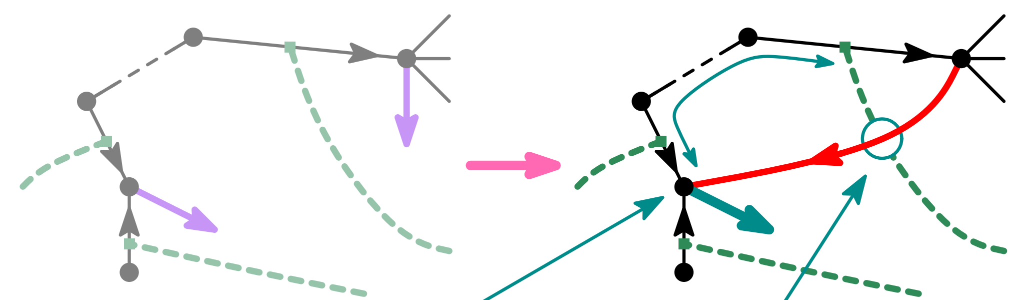

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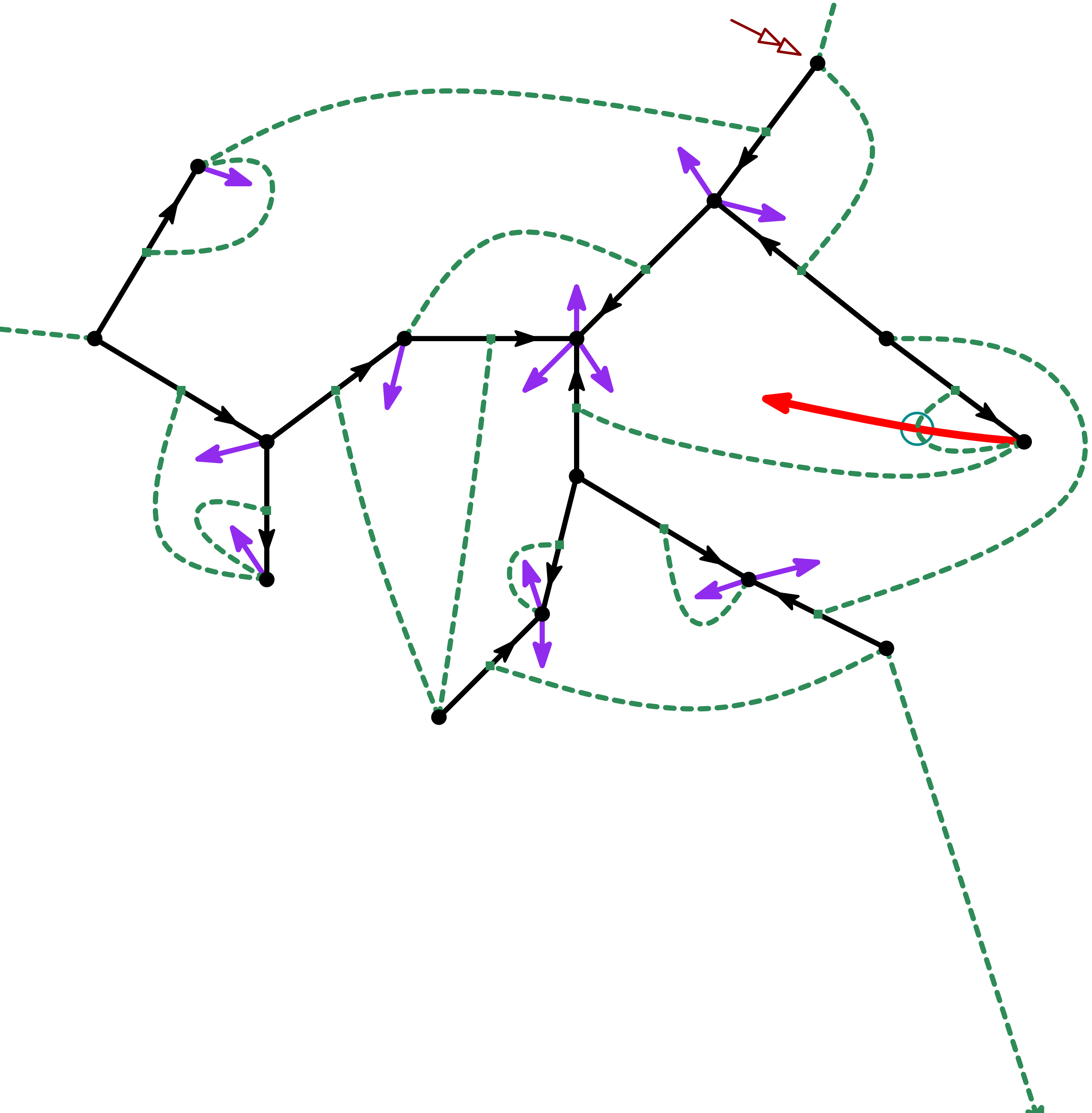

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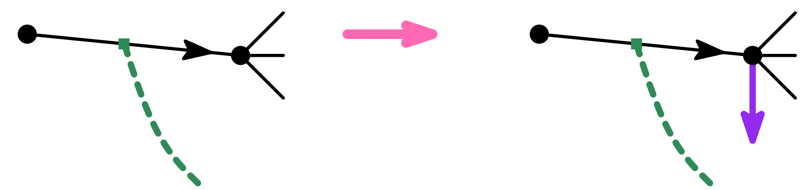

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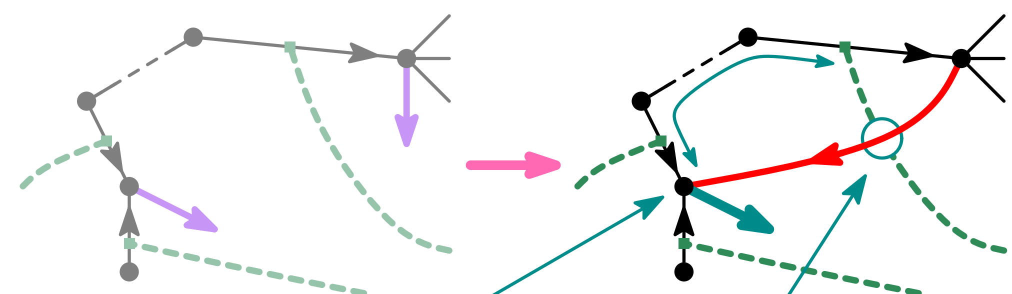

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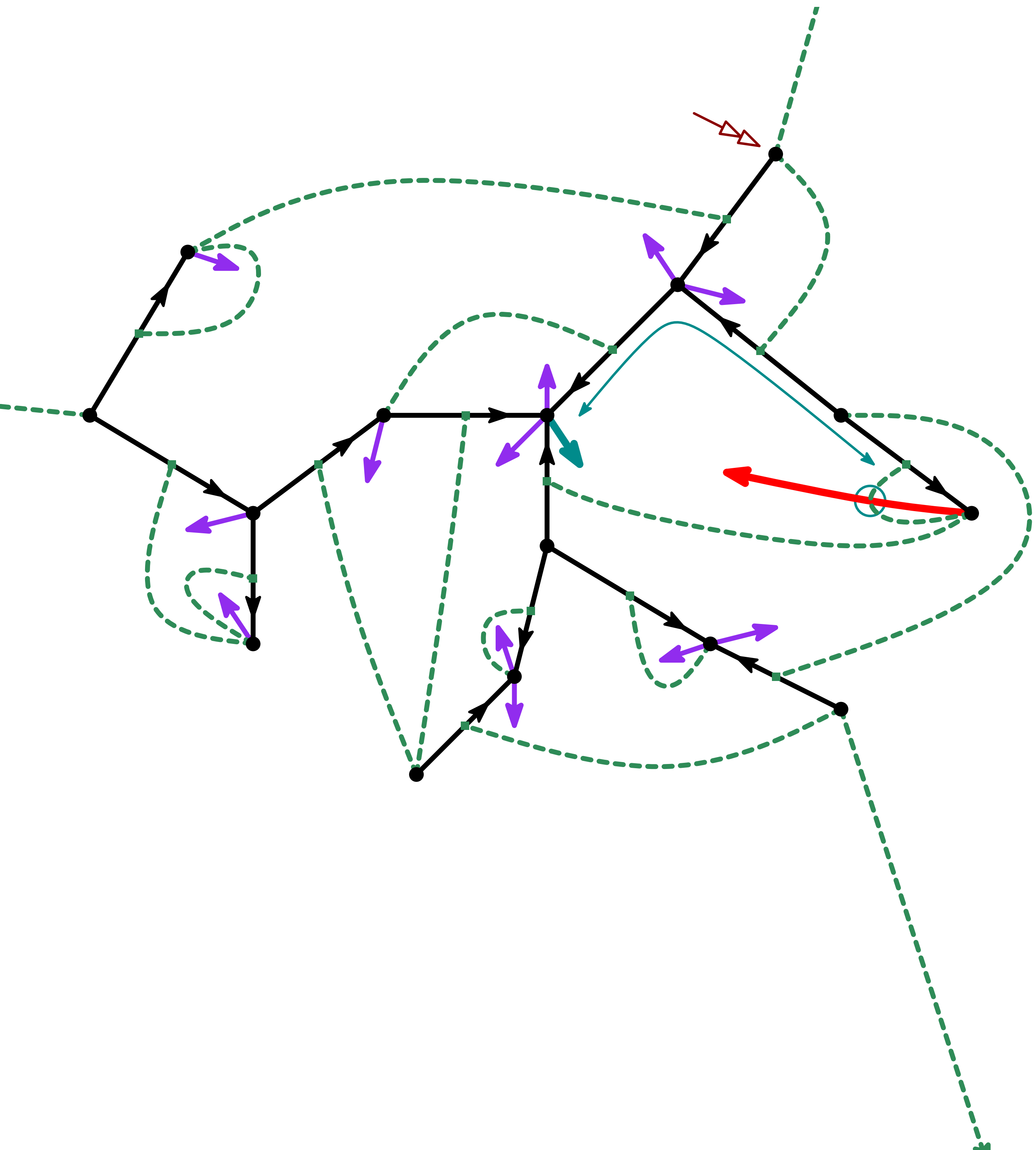

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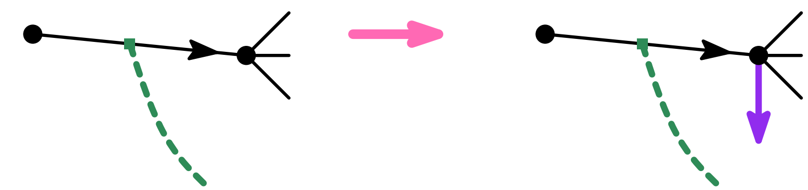

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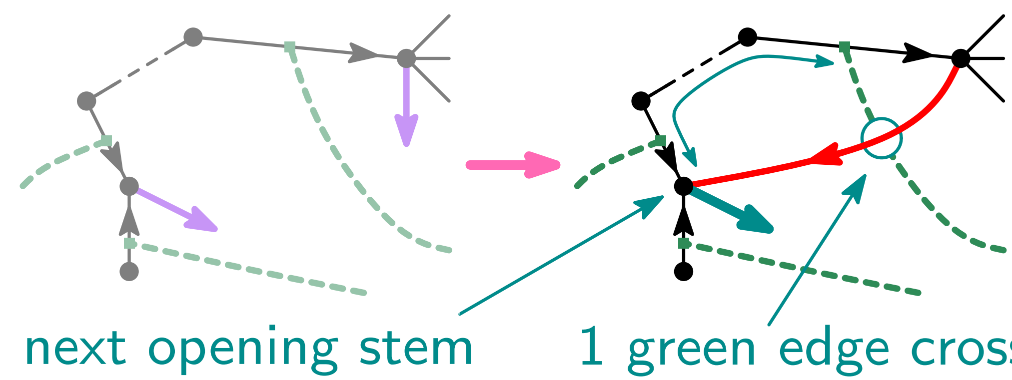

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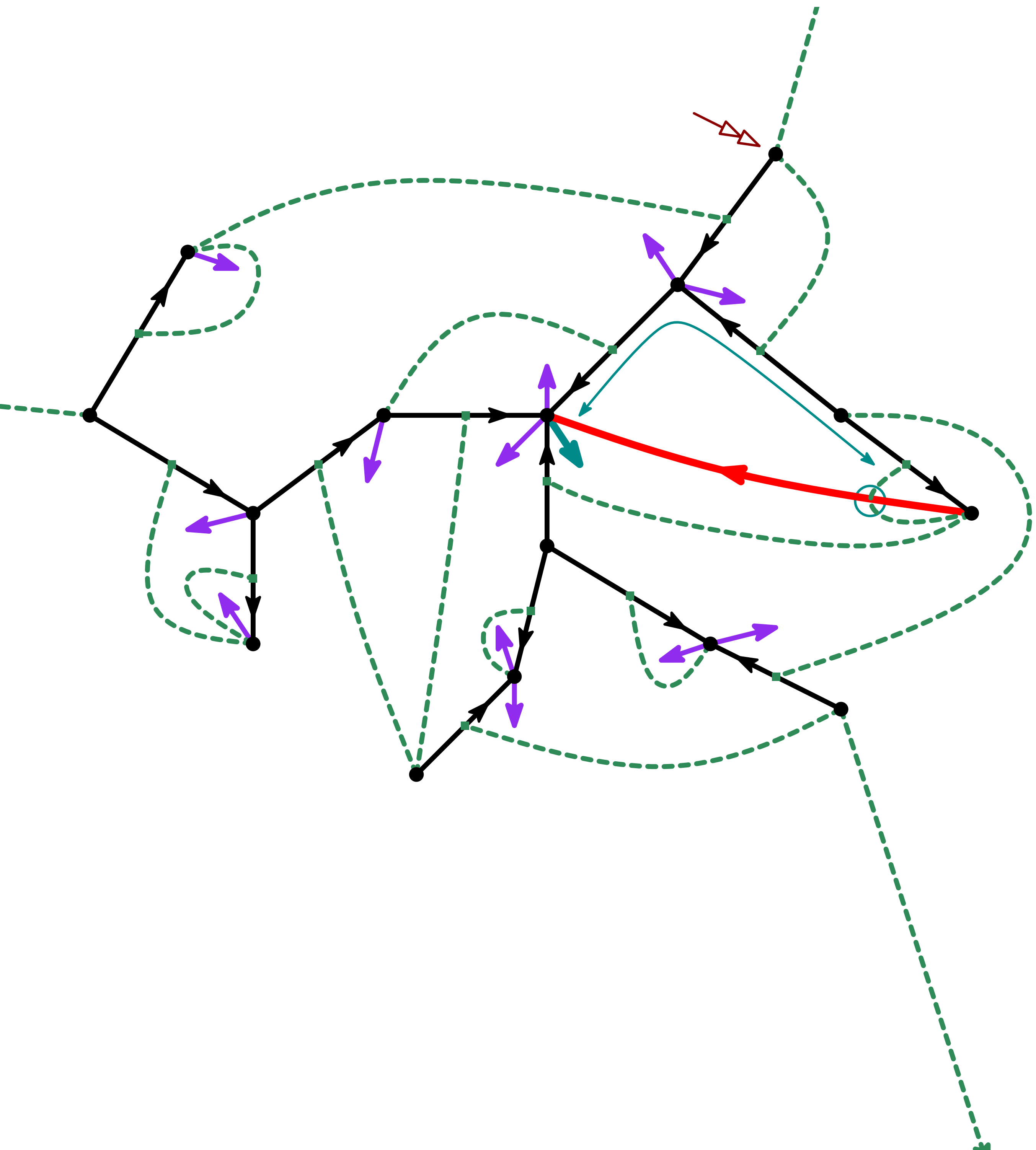

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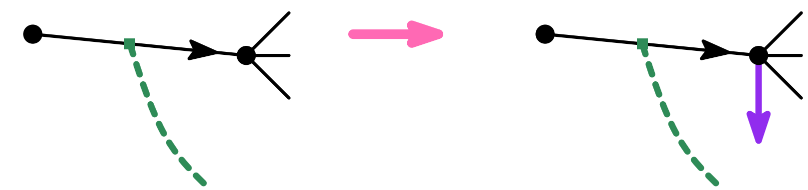

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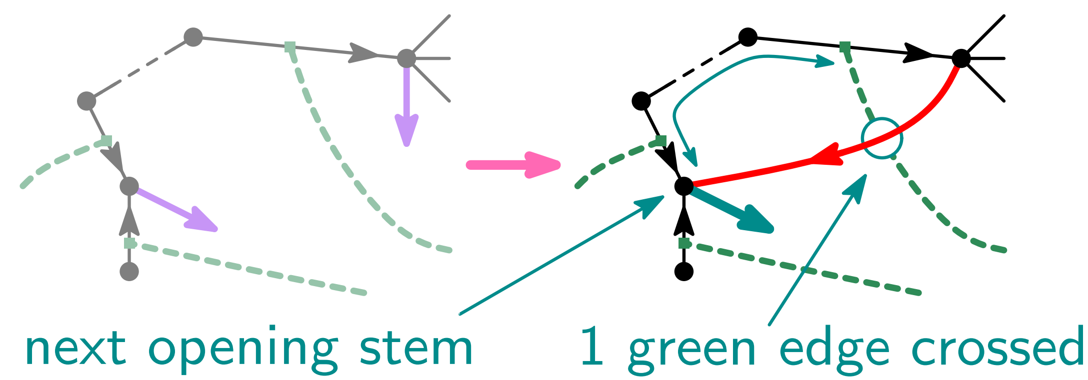

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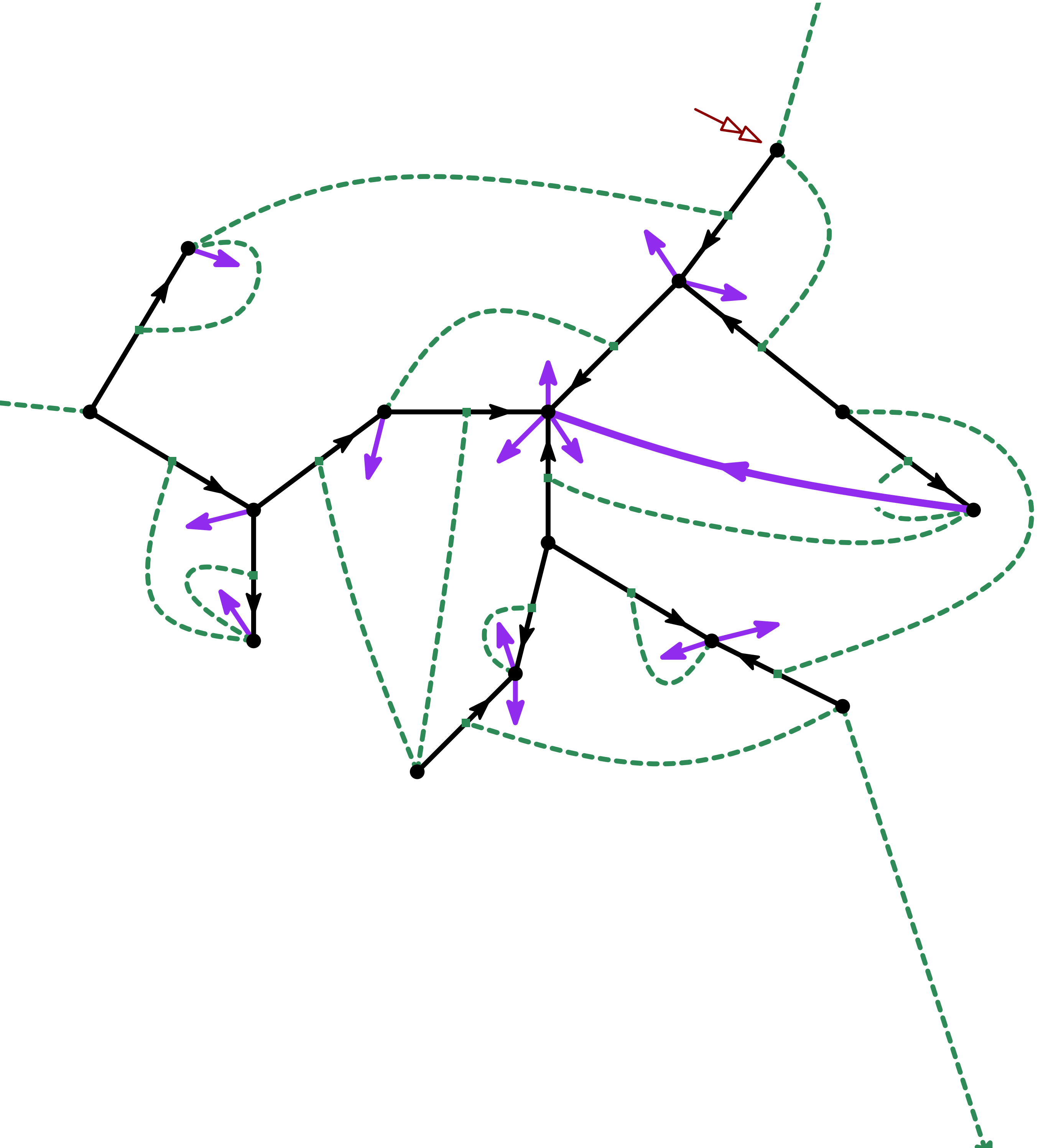

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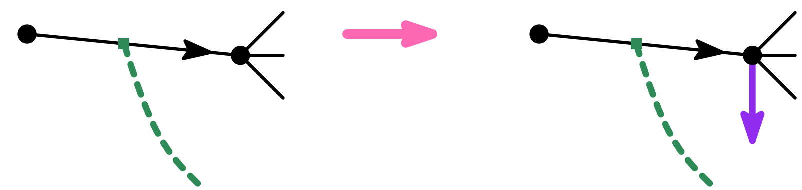

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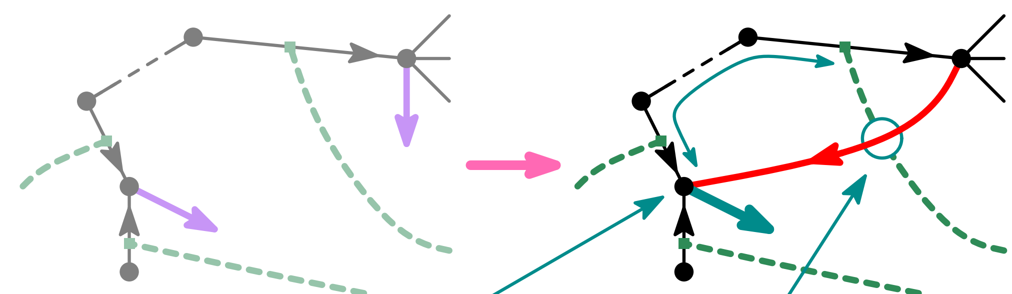

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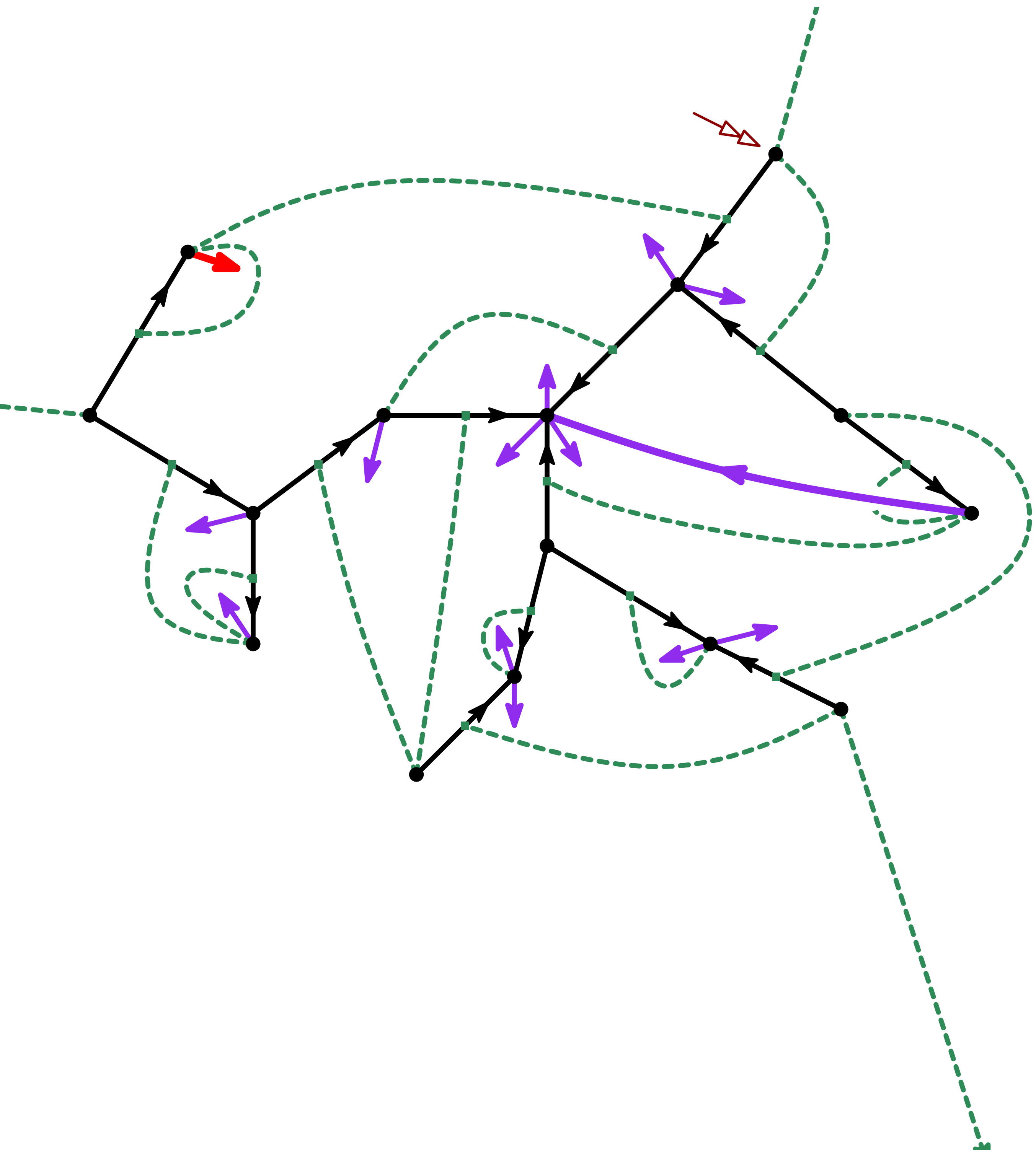

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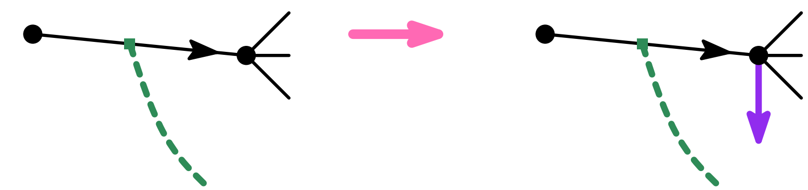

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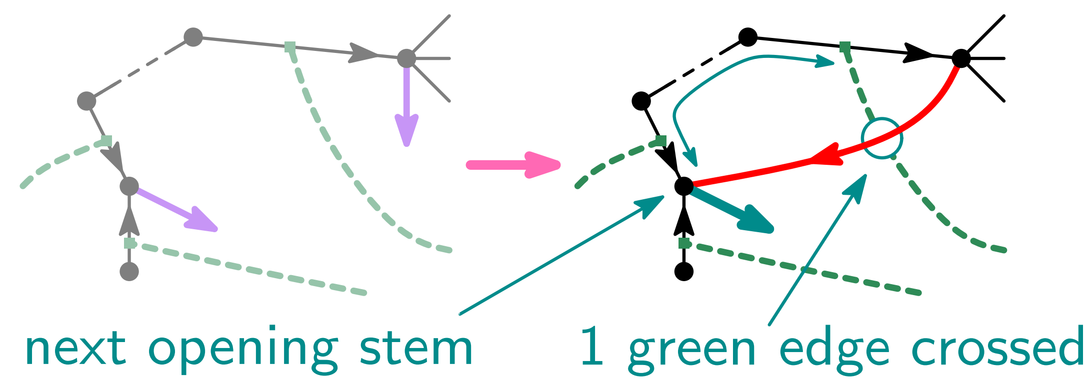
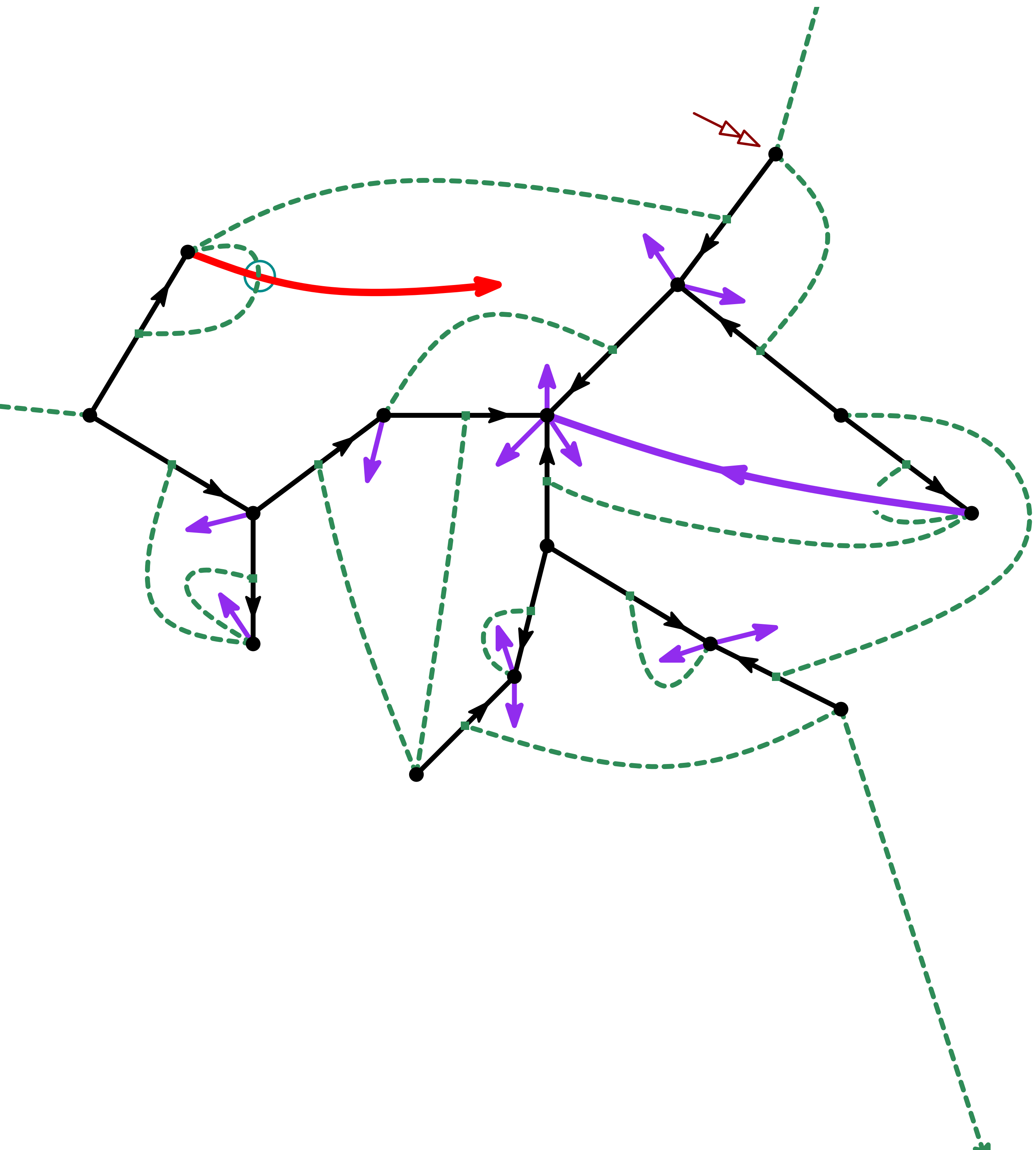
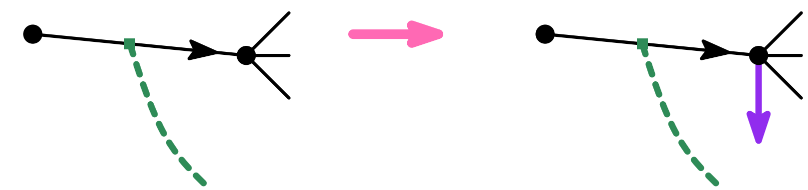


From bipartite cubic maps to simple maps



- Apply the following local rule :



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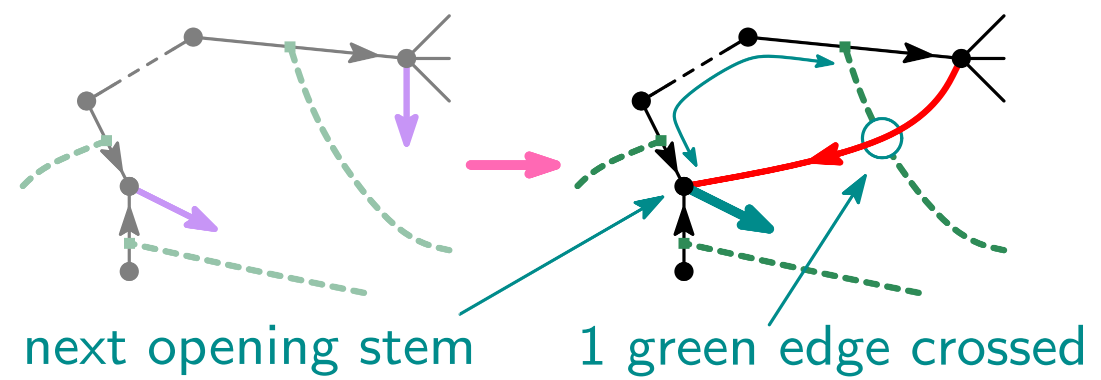

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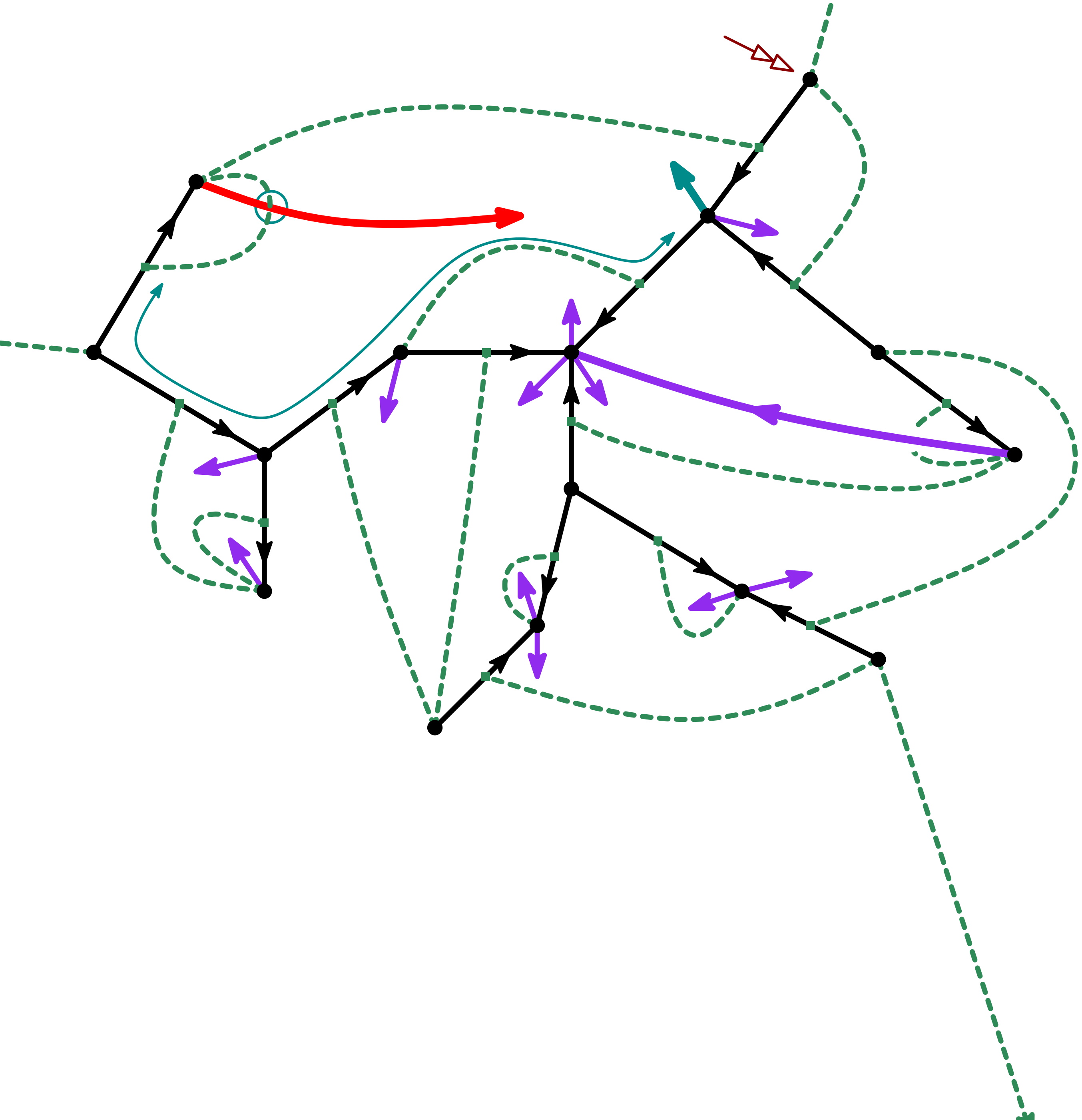

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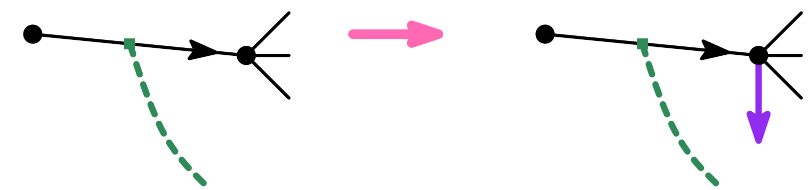

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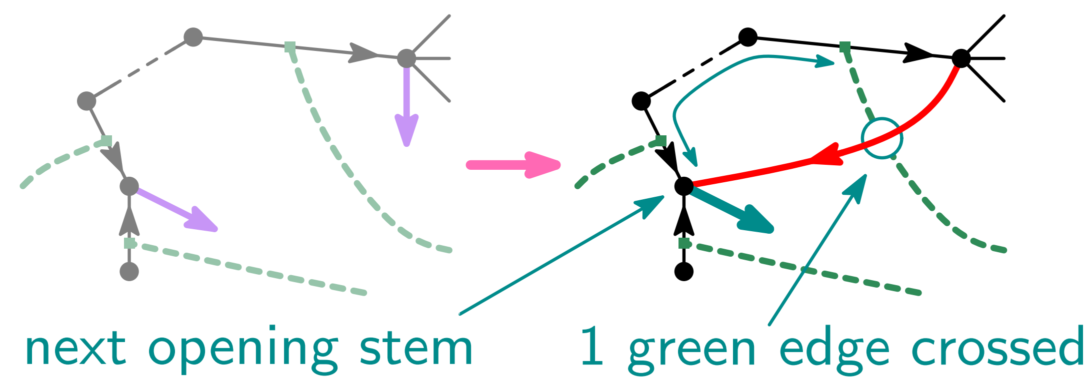

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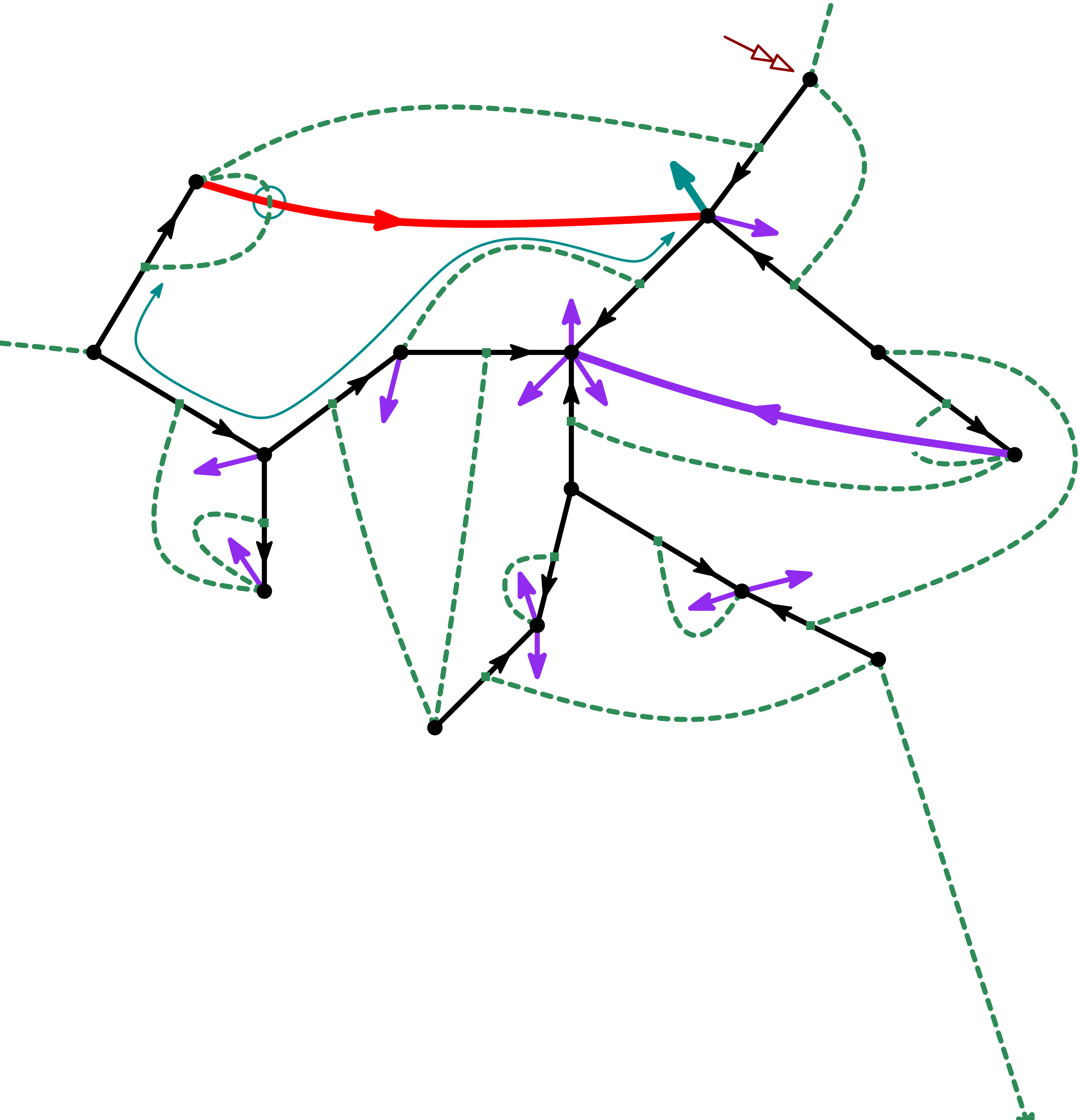

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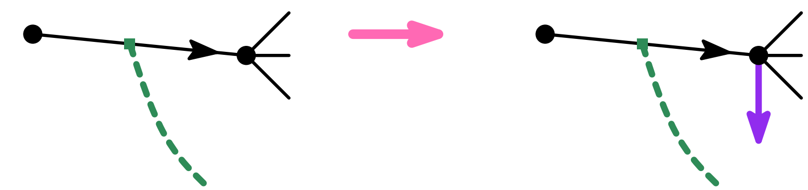

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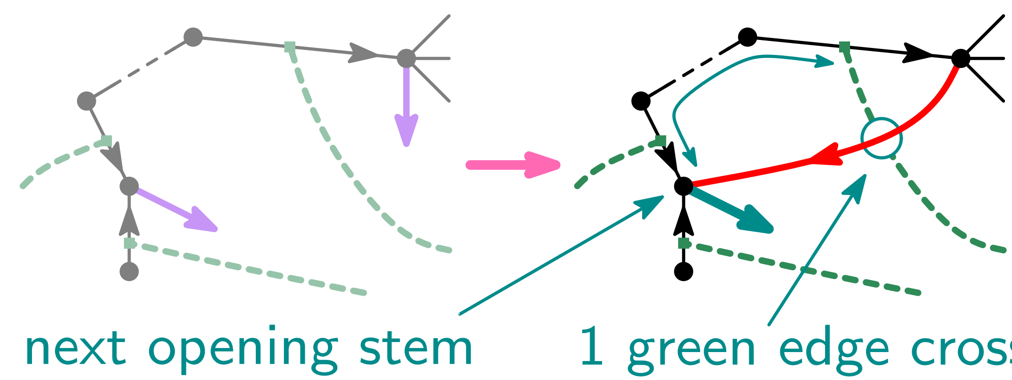

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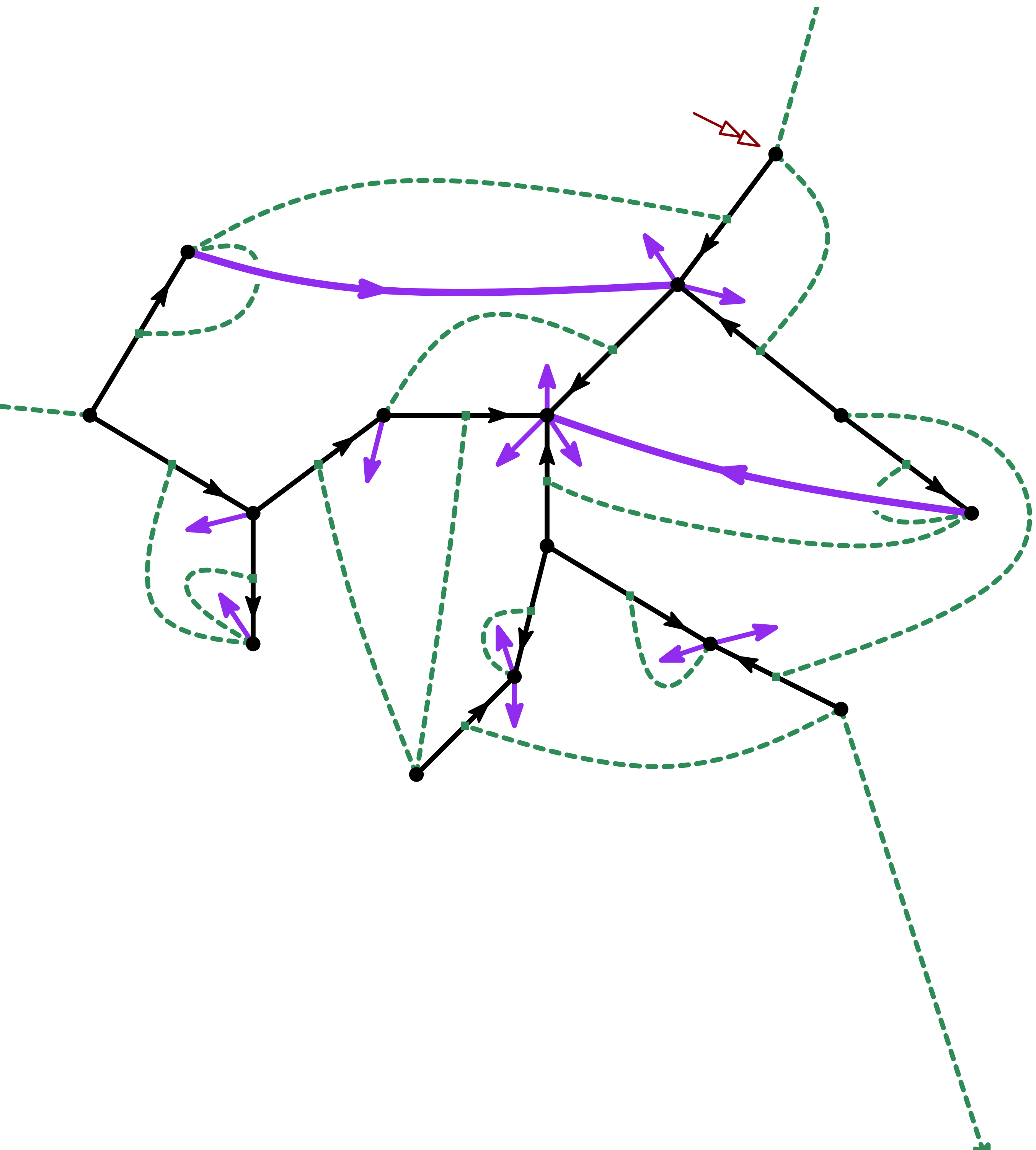

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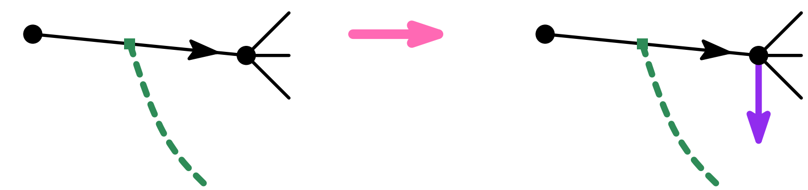

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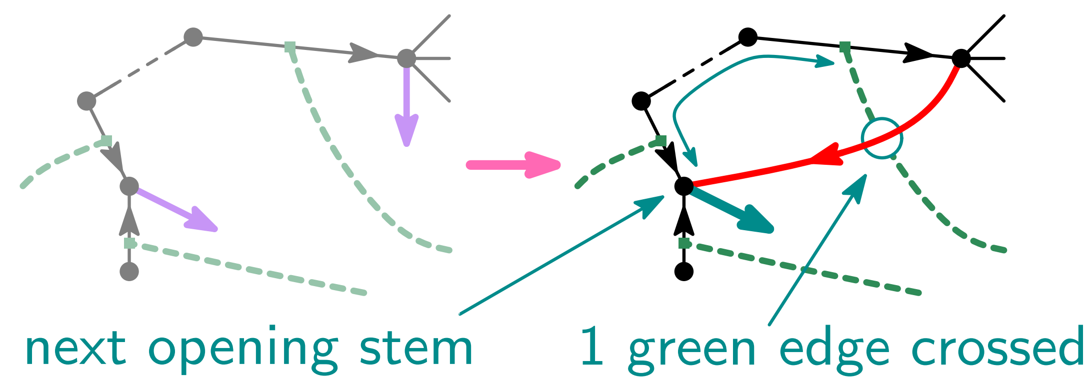

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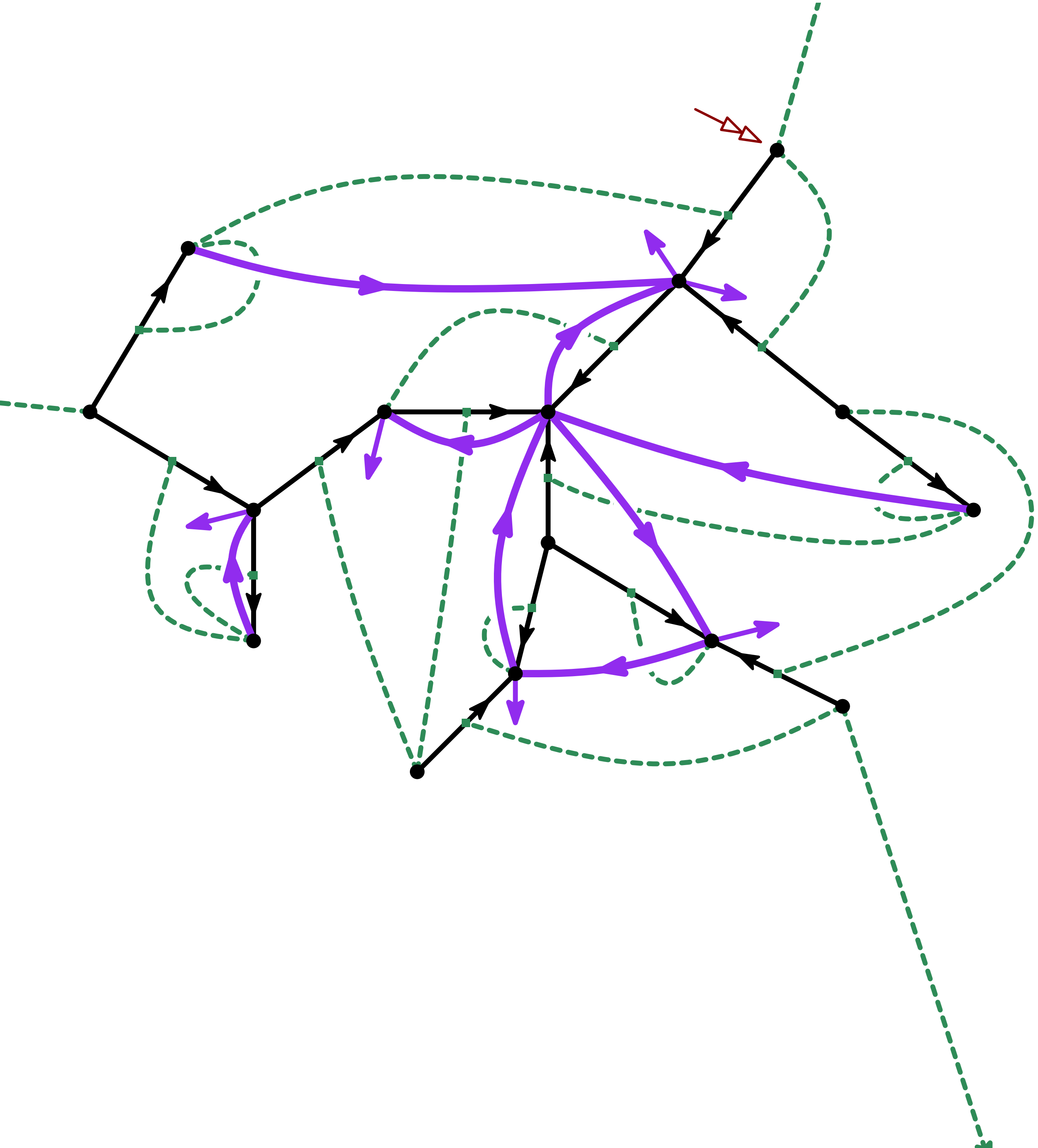

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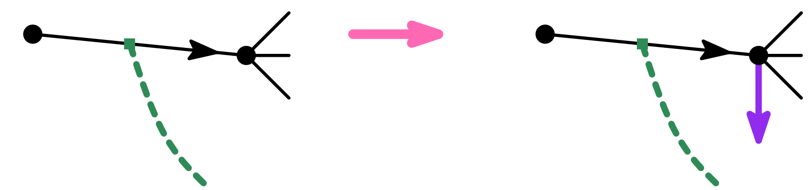

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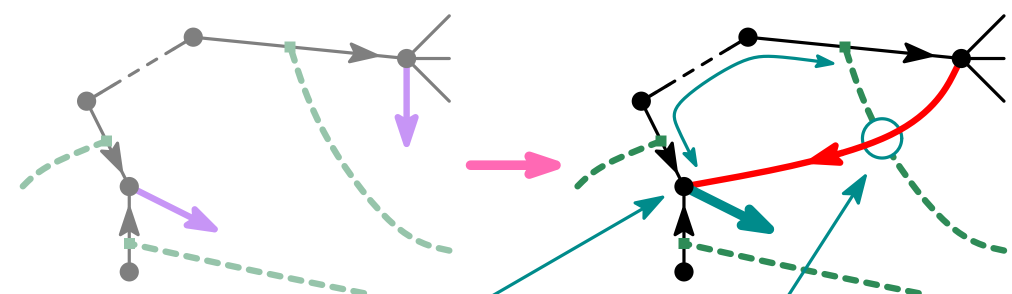

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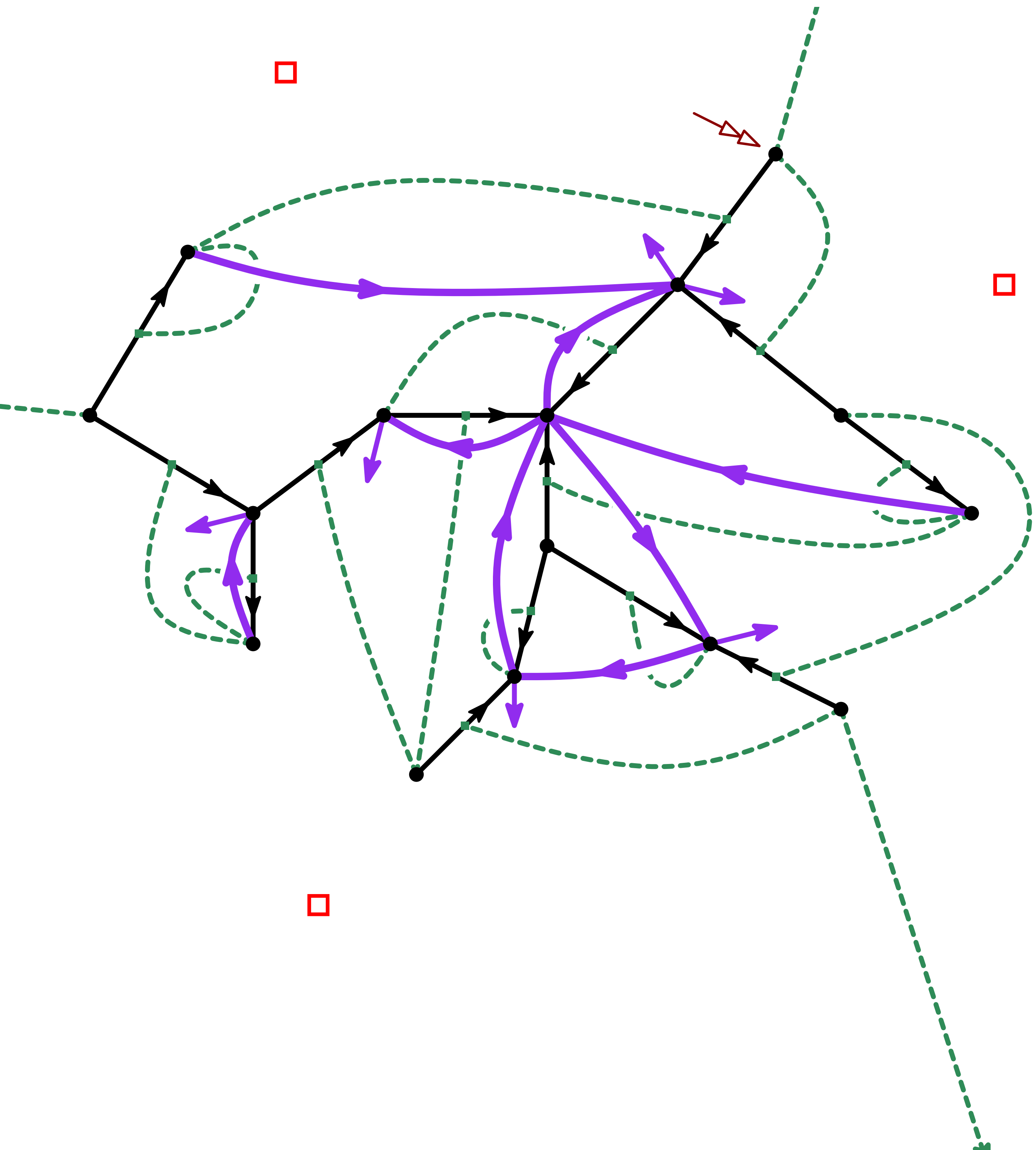

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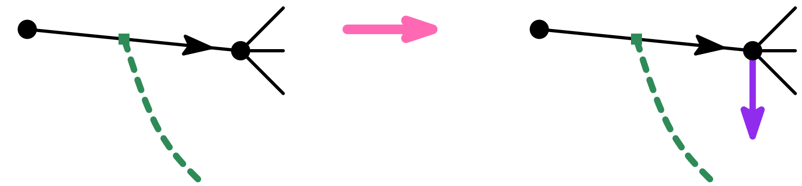
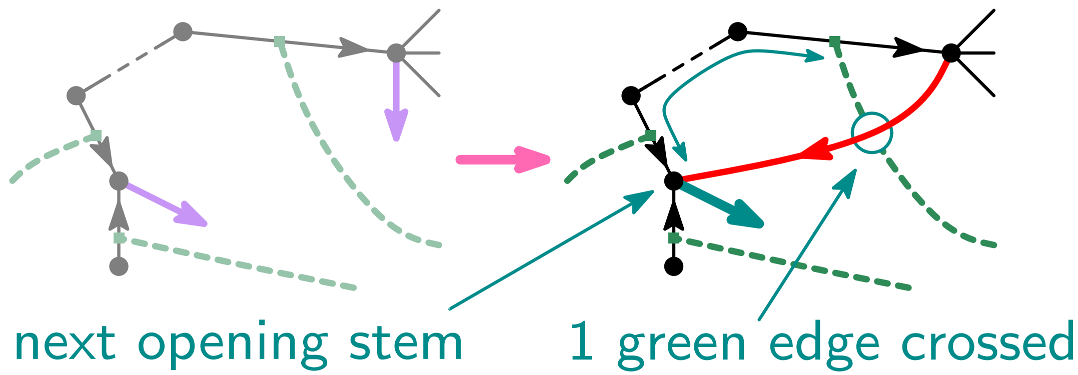
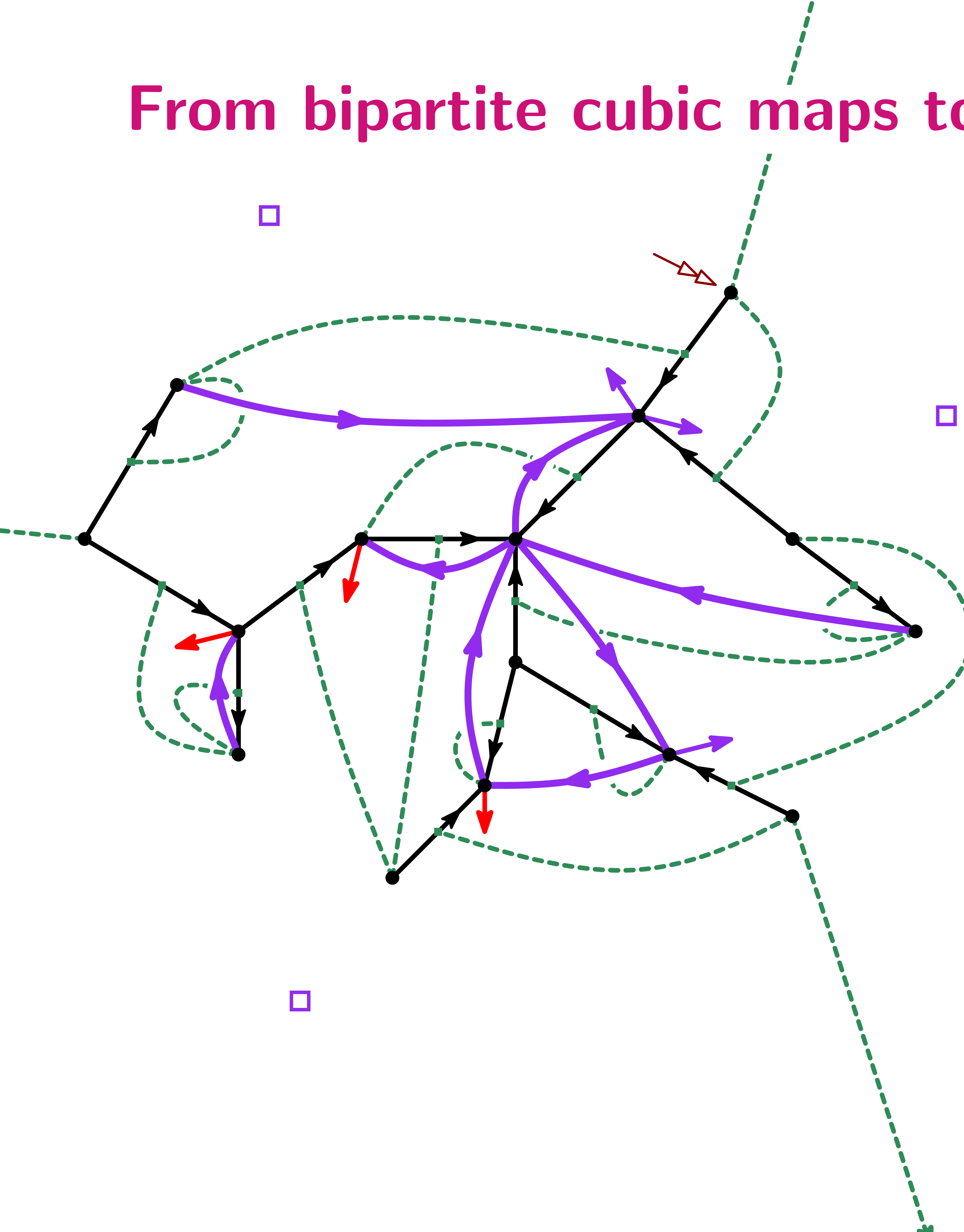


- Turning clockwise around the tree, do the following closures:

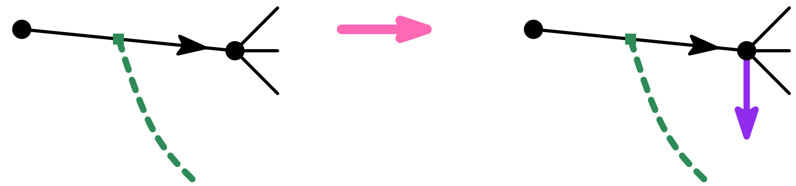


- Add 3 vertices and close the remaining opening stems sector by sector

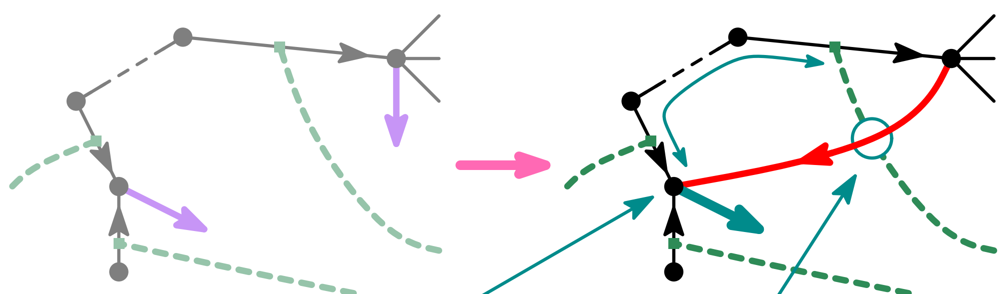
From bipartite cubic maps to simple maps



- Apply the following local rule :



- Turning clockwise around the tree, do the following closures:



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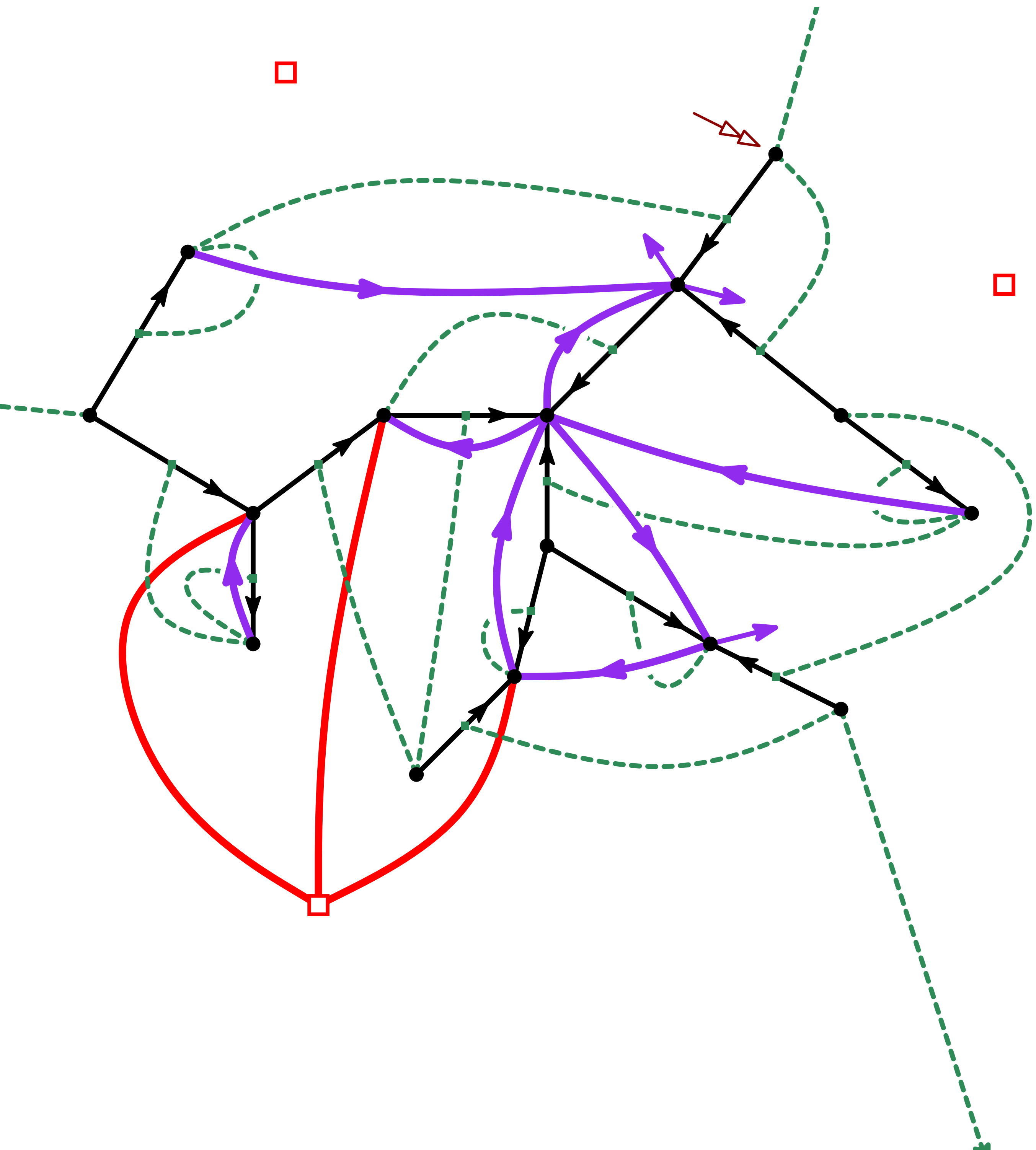
From bipartite cubic maps to simple maps



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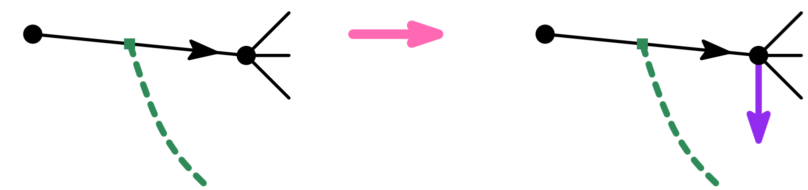

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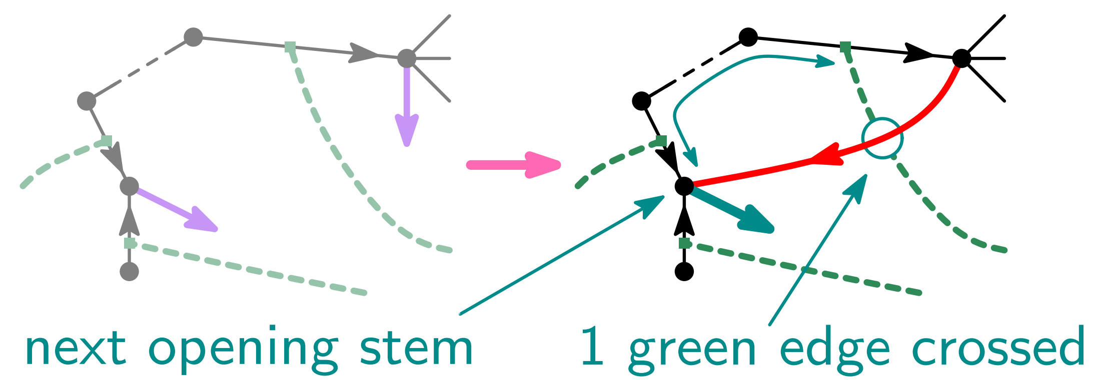

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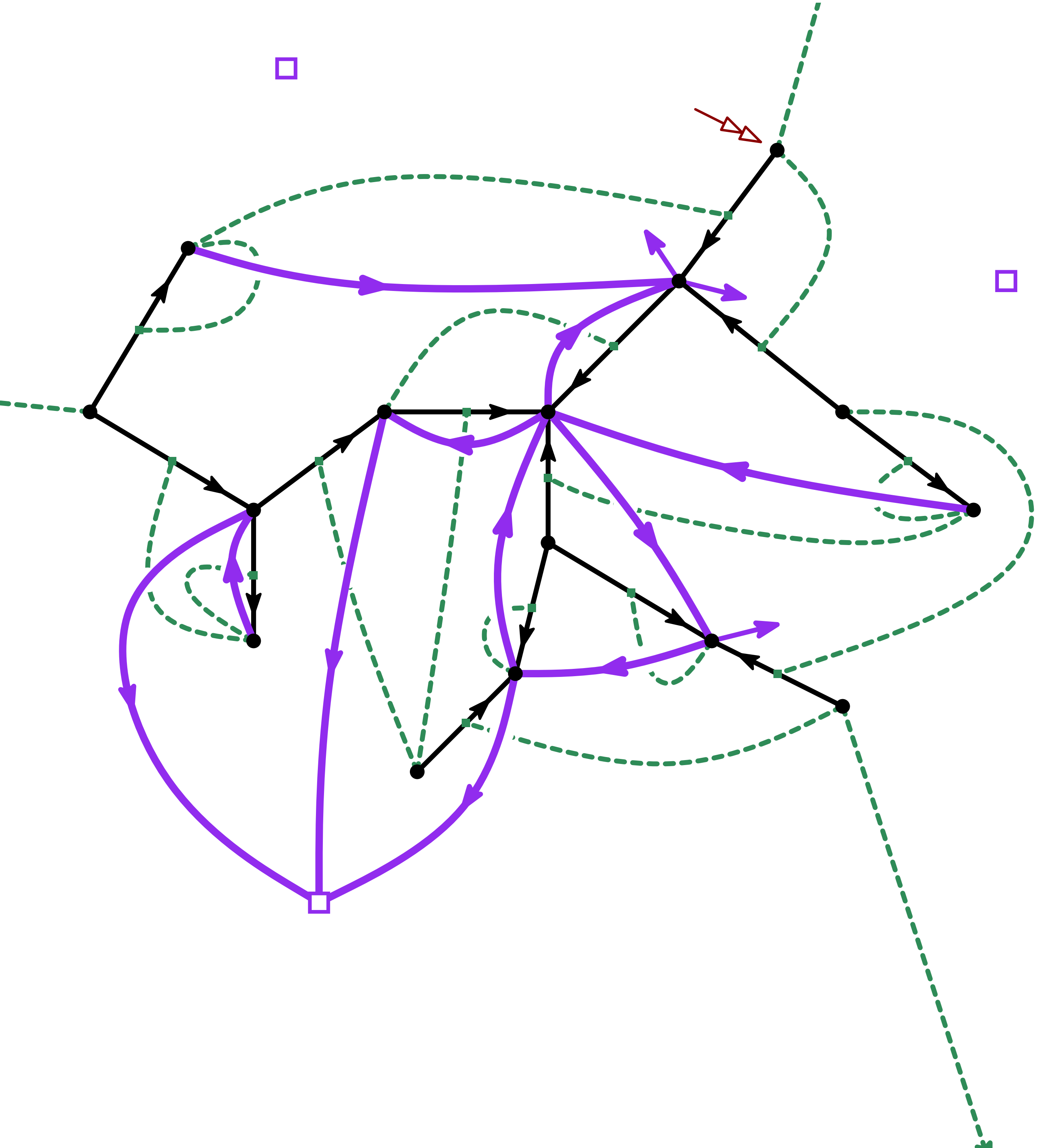
From bipartite cubic maps to simple maps



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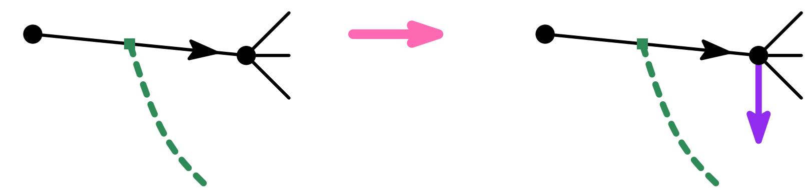

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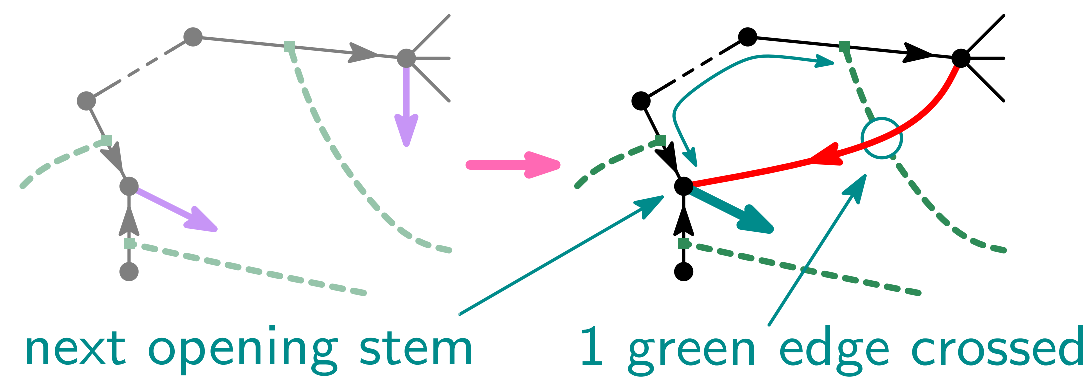

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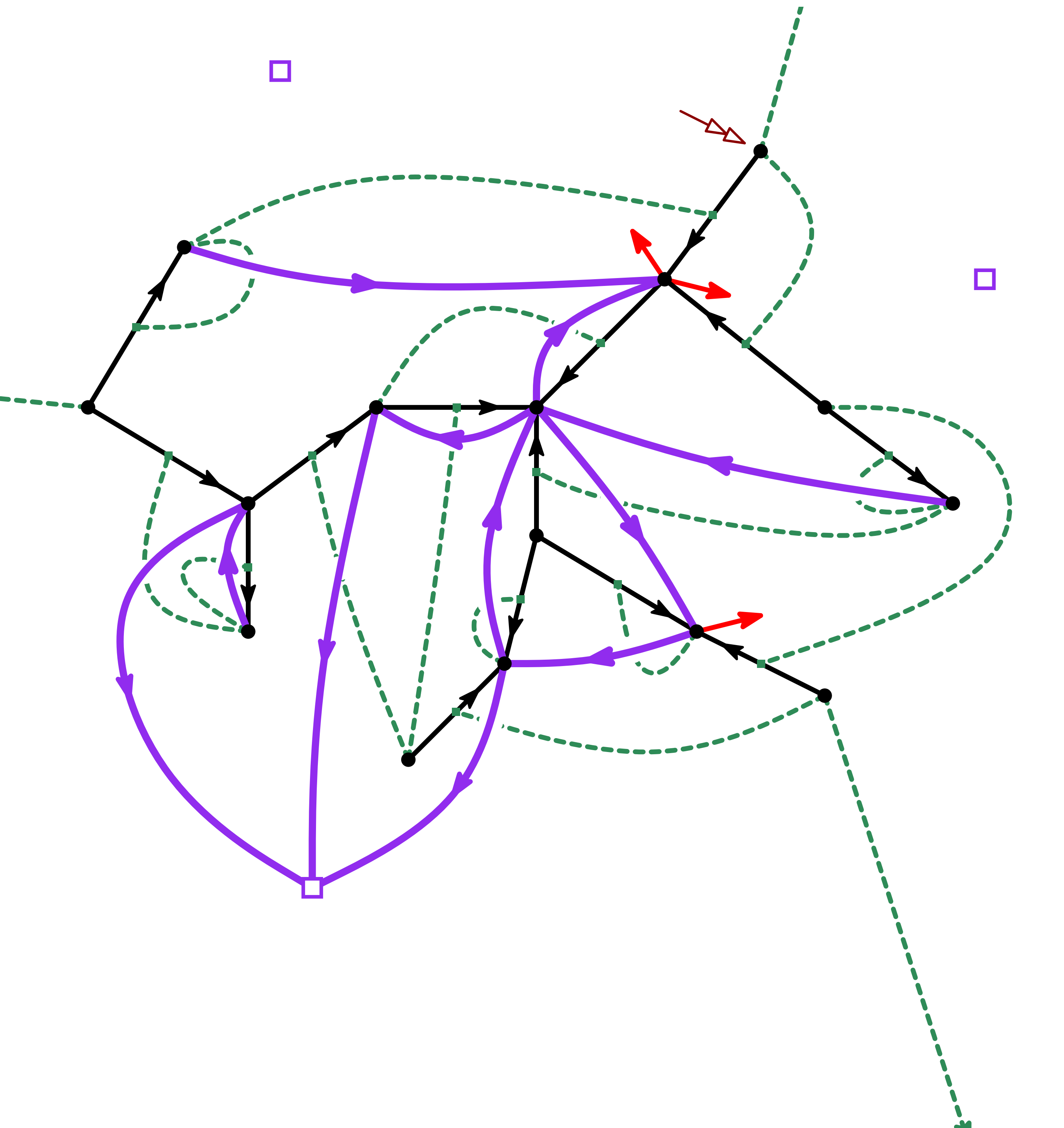
From bipartite cubic maps to simple maps



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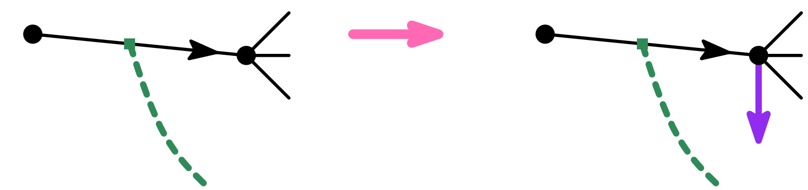

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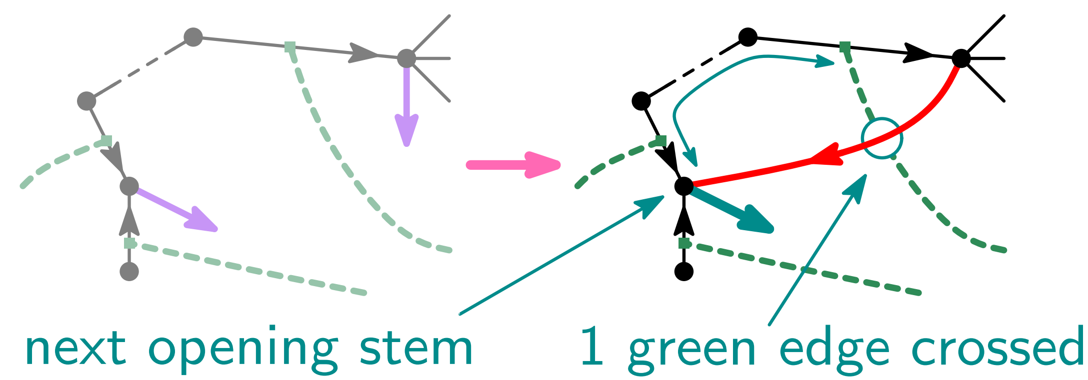

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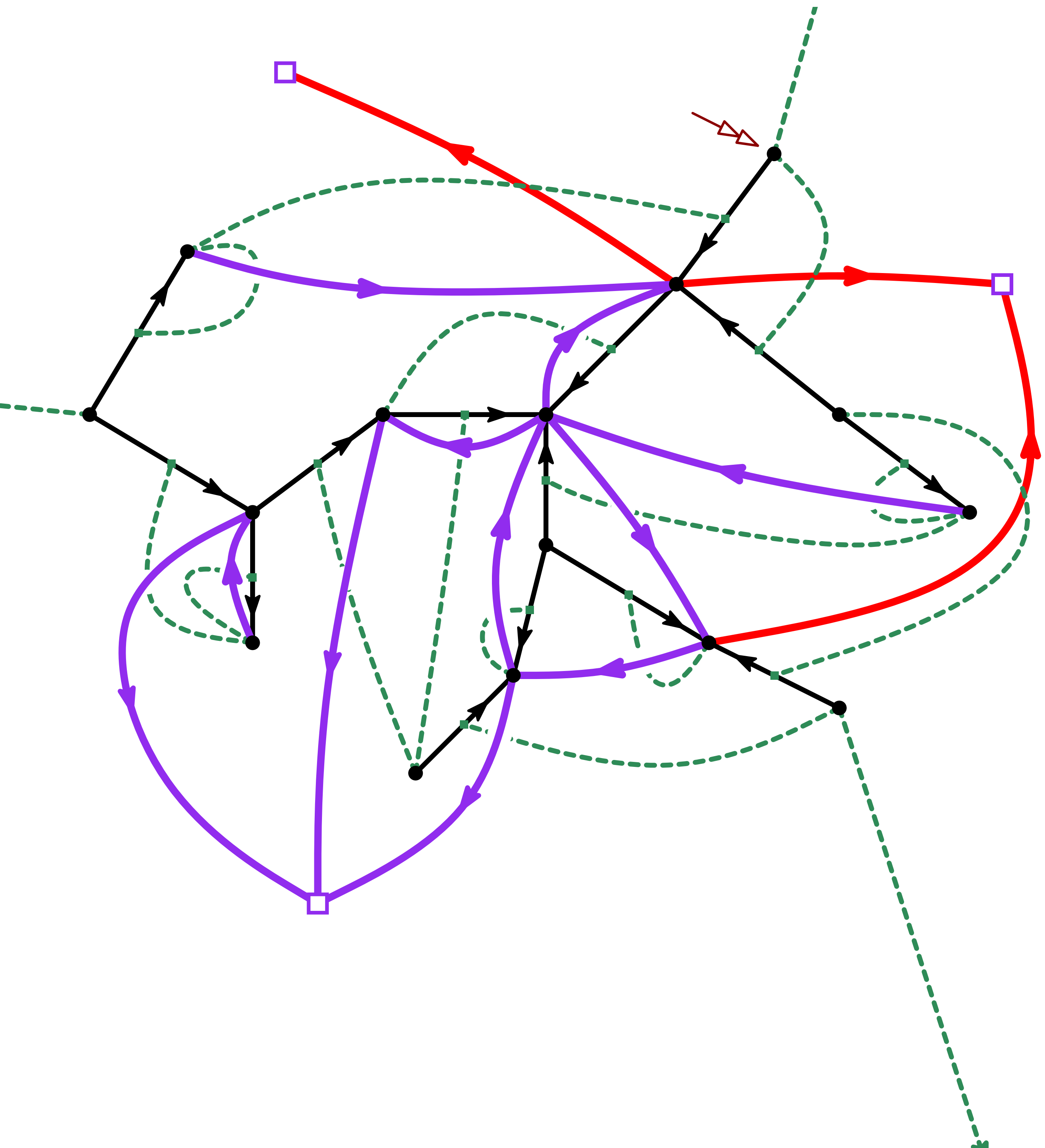
From bipartite cubic maps to simple maps



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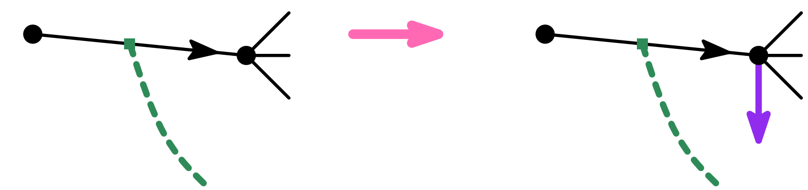

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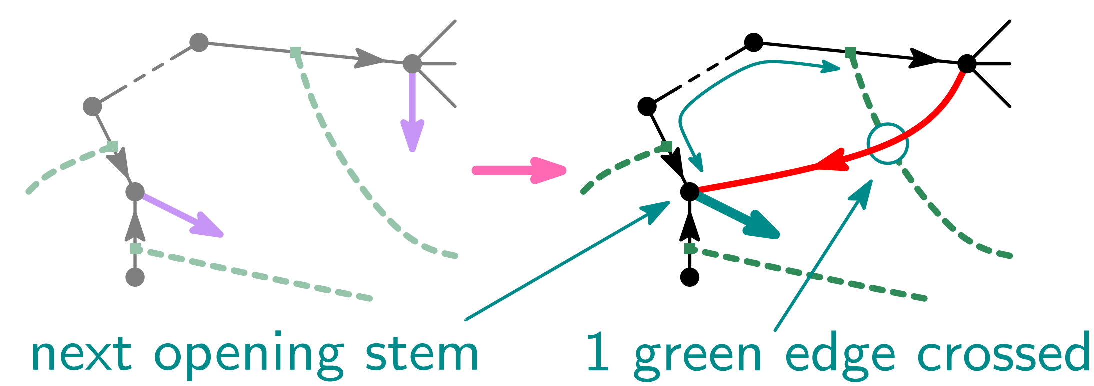

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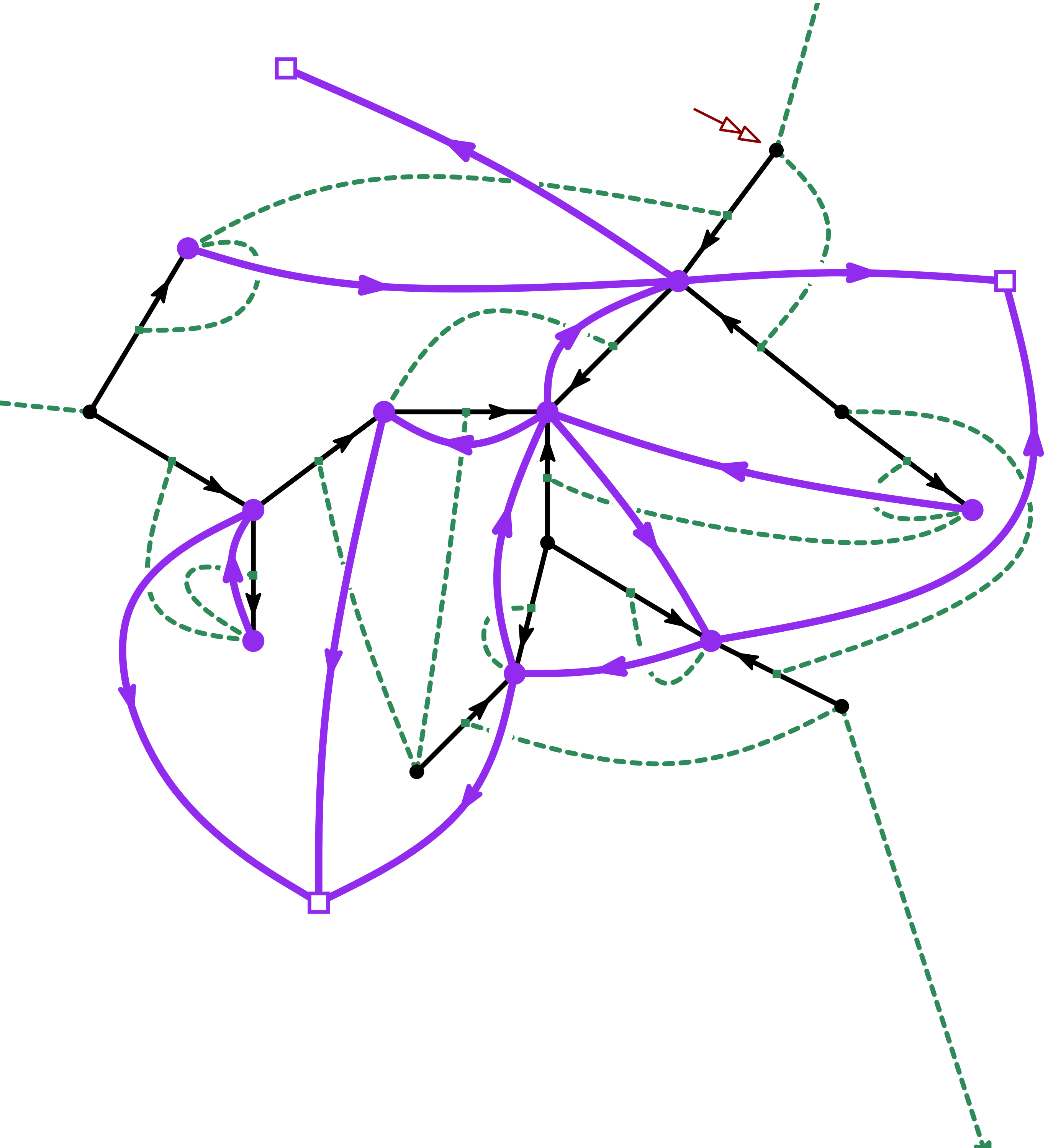
From bipartite cubic maps to simple maps



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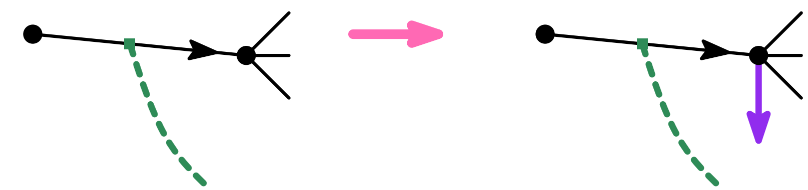

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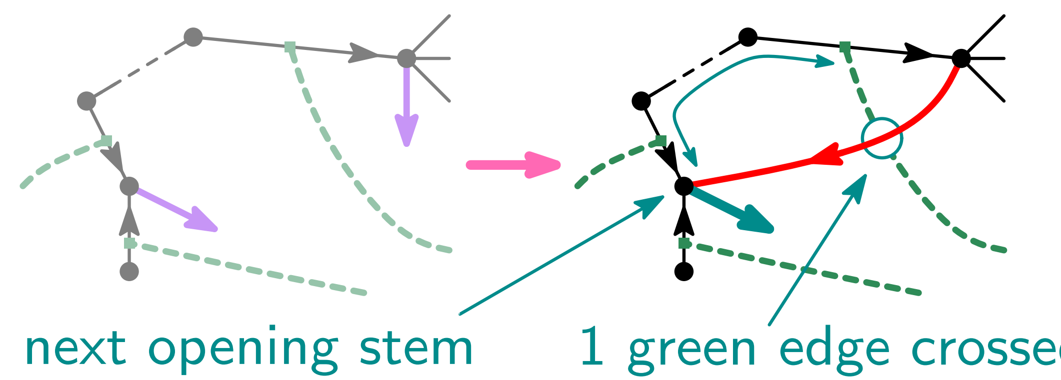

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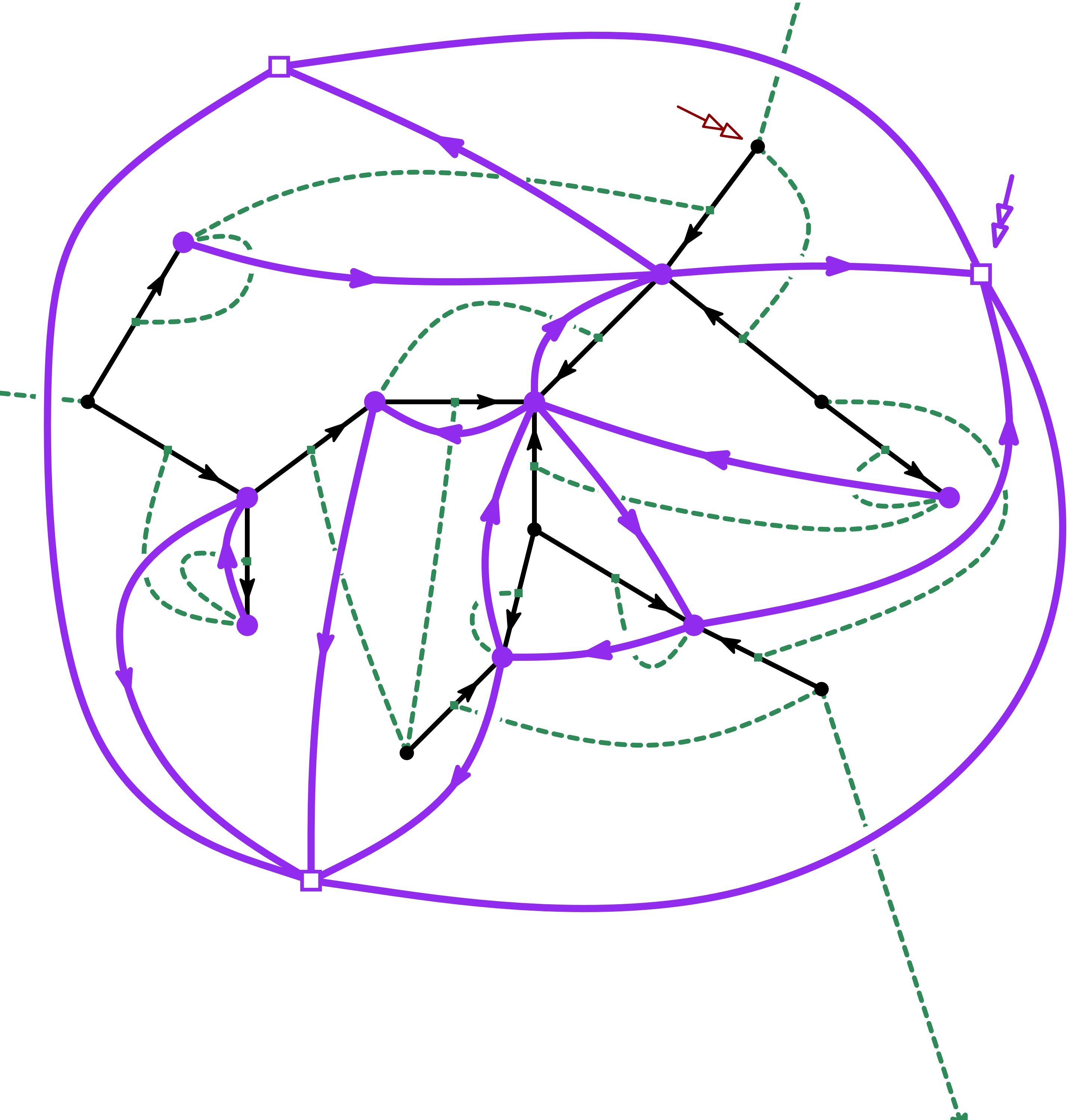
From bipartite cubic maps to simple maps



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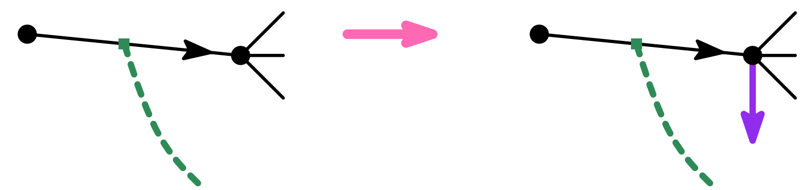

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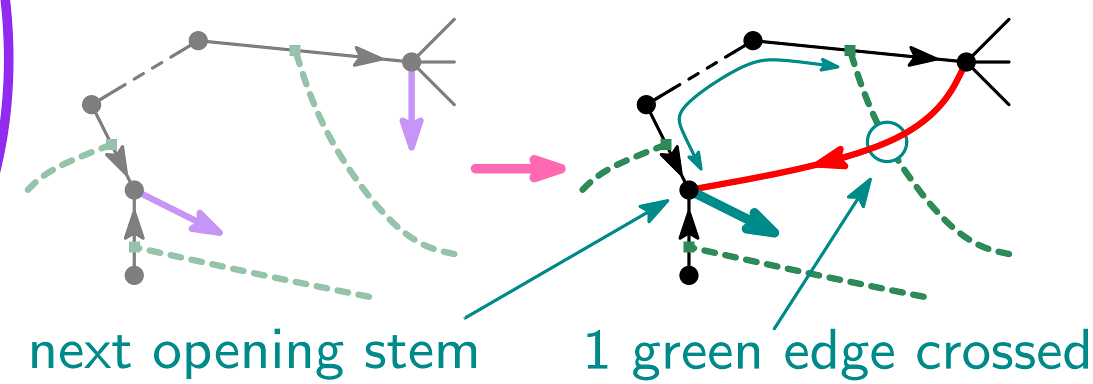
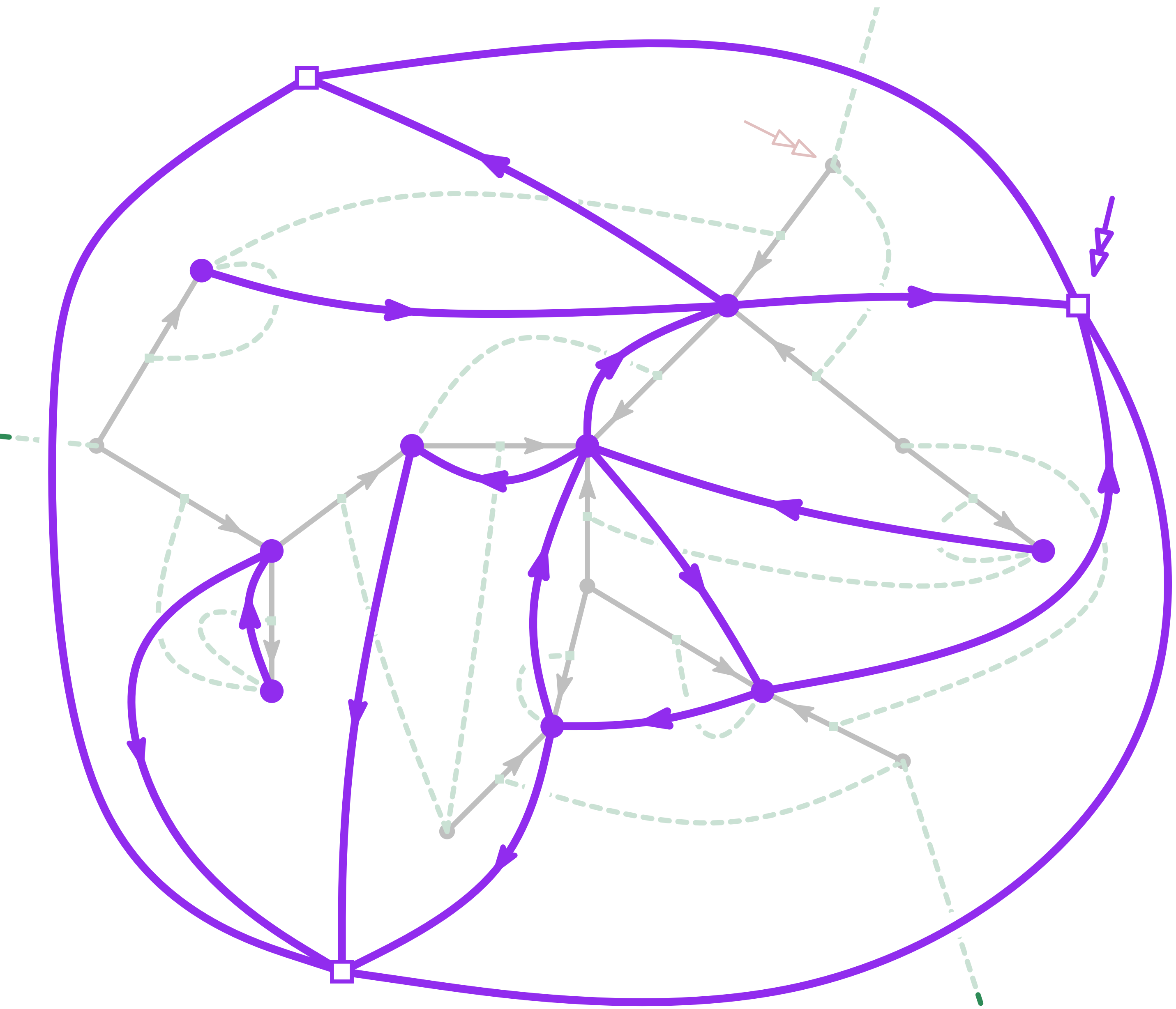


- Add 3 vertices and close the remaining opening stems sector by sector
- Connect the 3 outer vertices into a triangle

From bipartite cubic maps to simple maps



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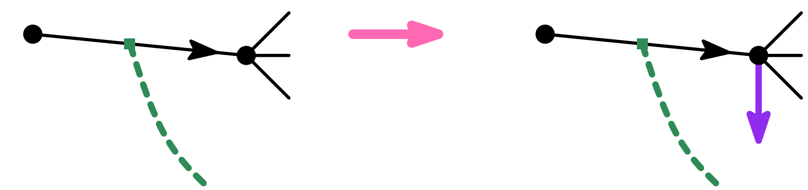

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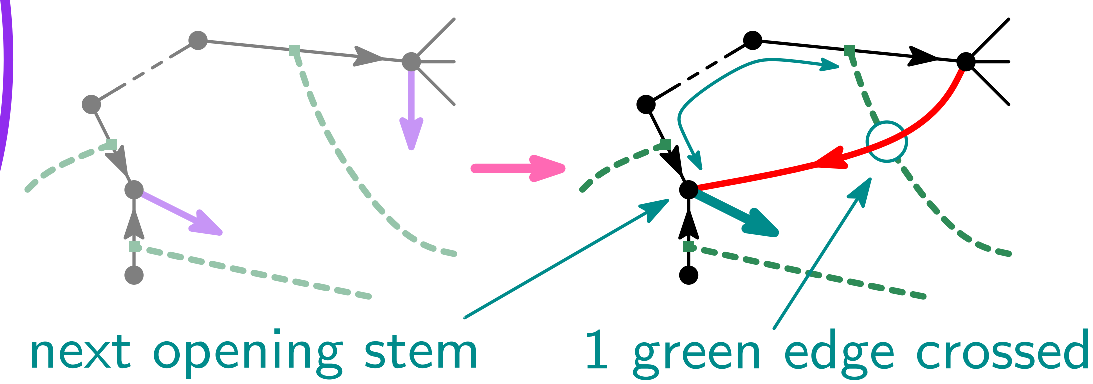
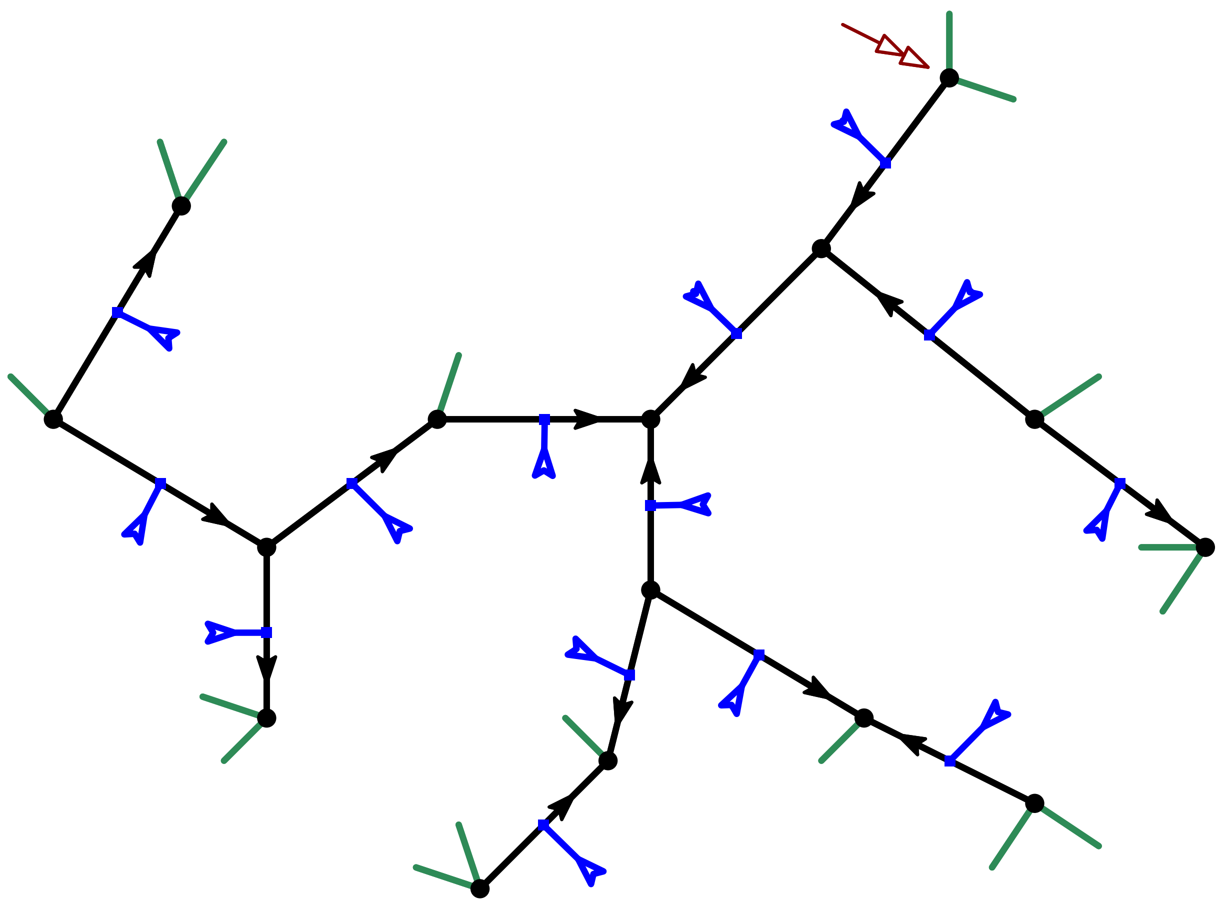


- Add 3 vertices and close the remaining opening stems sector by sector

Theorem : [ABCF] This is a bijection between **outer-triangular simple maps** and **bipartite cubic maps**

From bipartite cubic maps to simple maps



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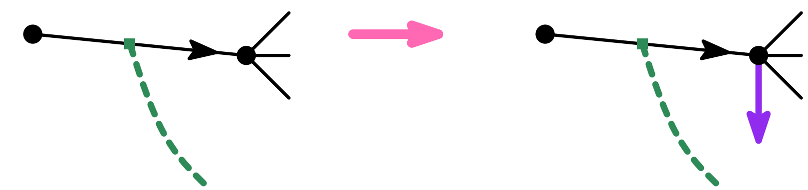

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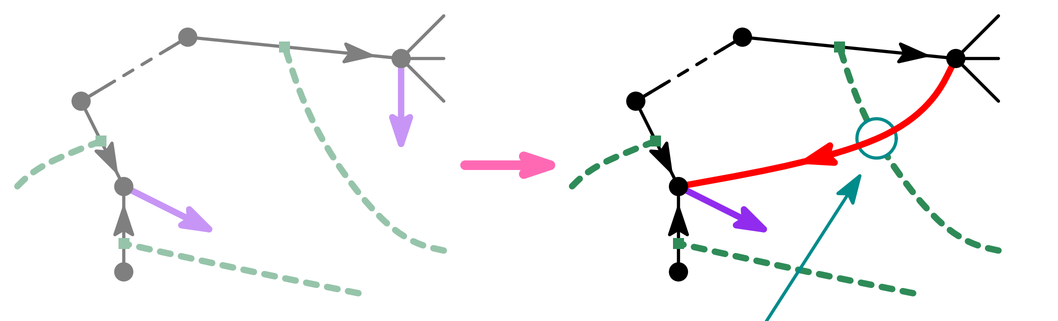

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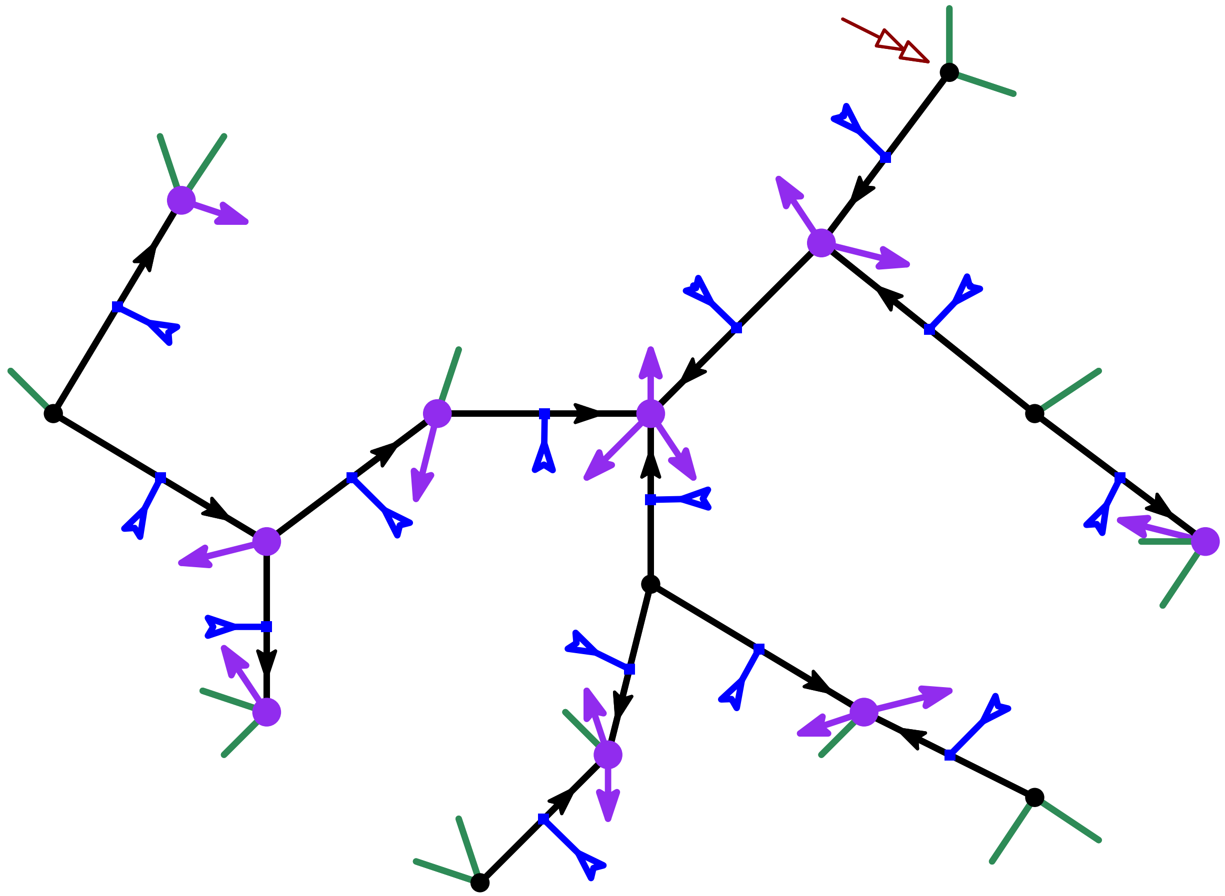

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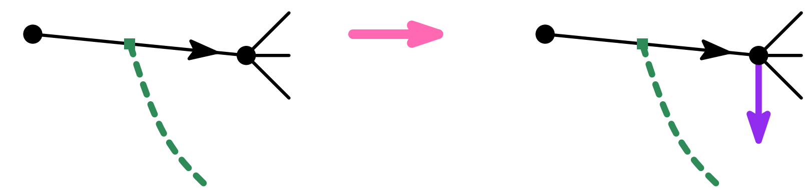

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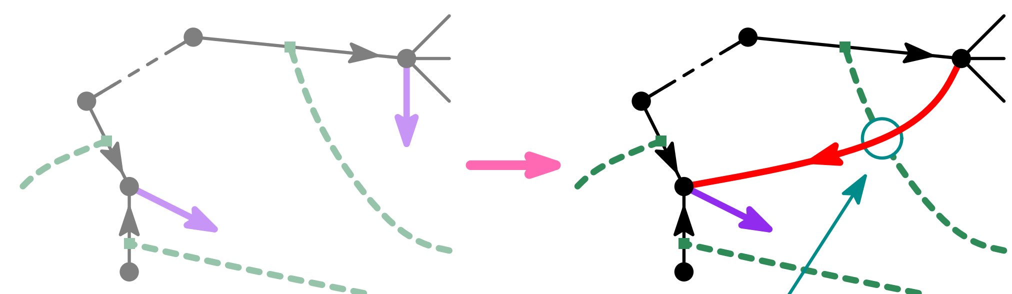

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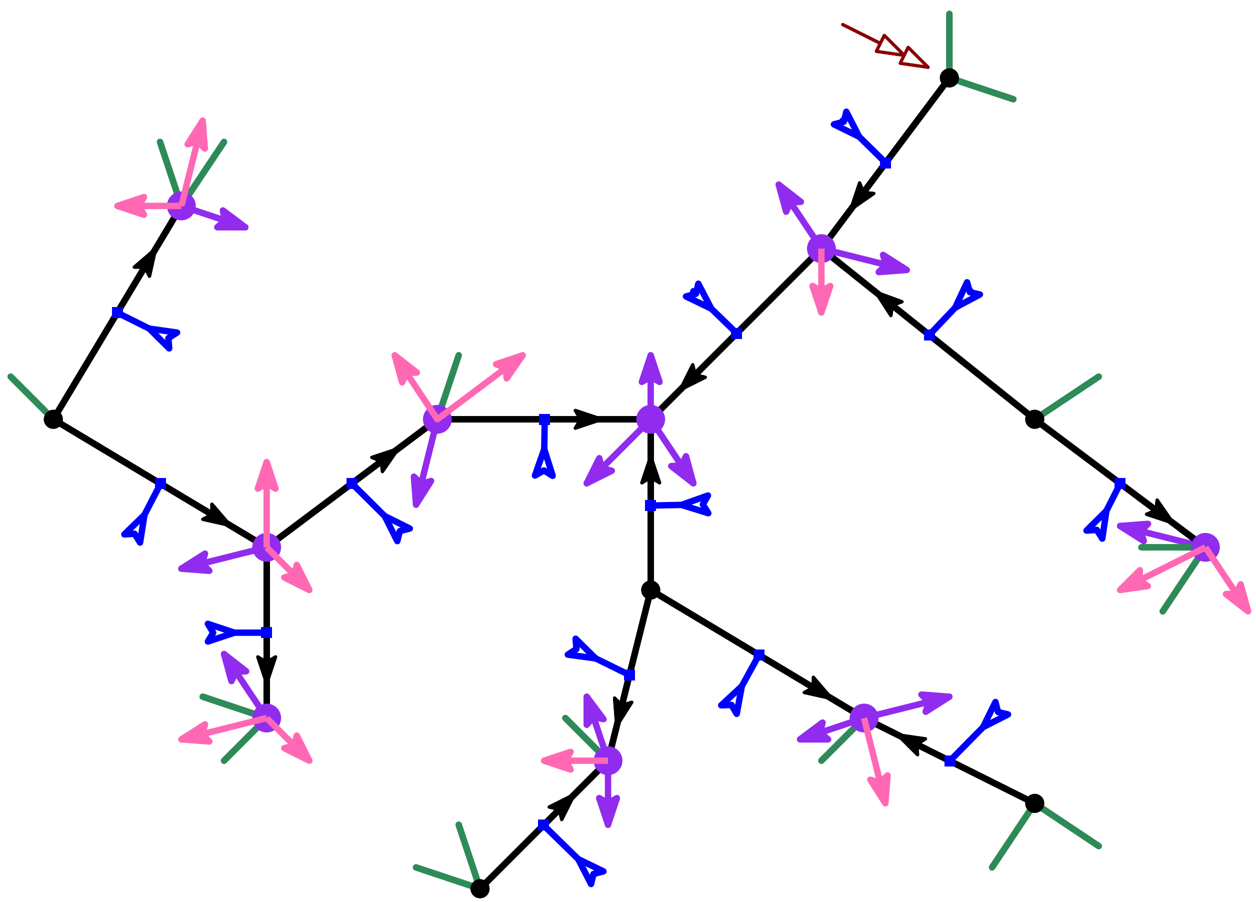

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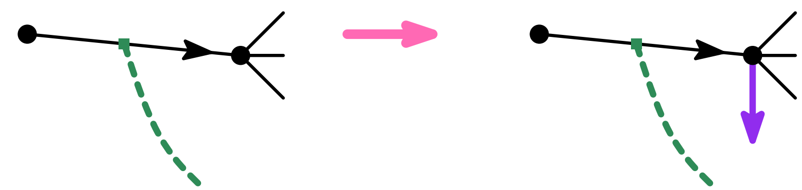

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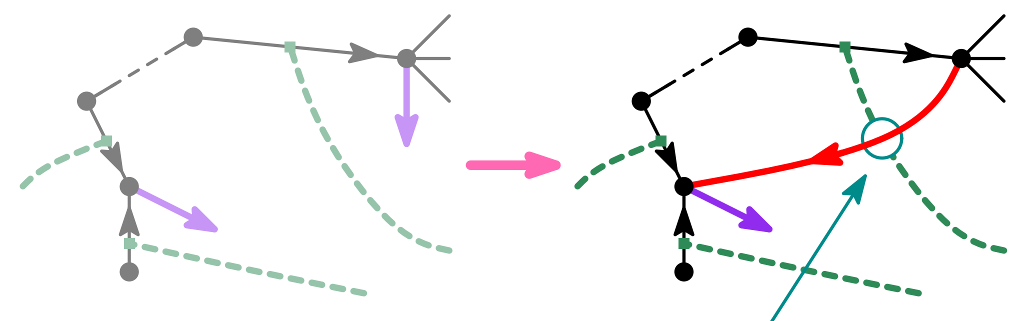

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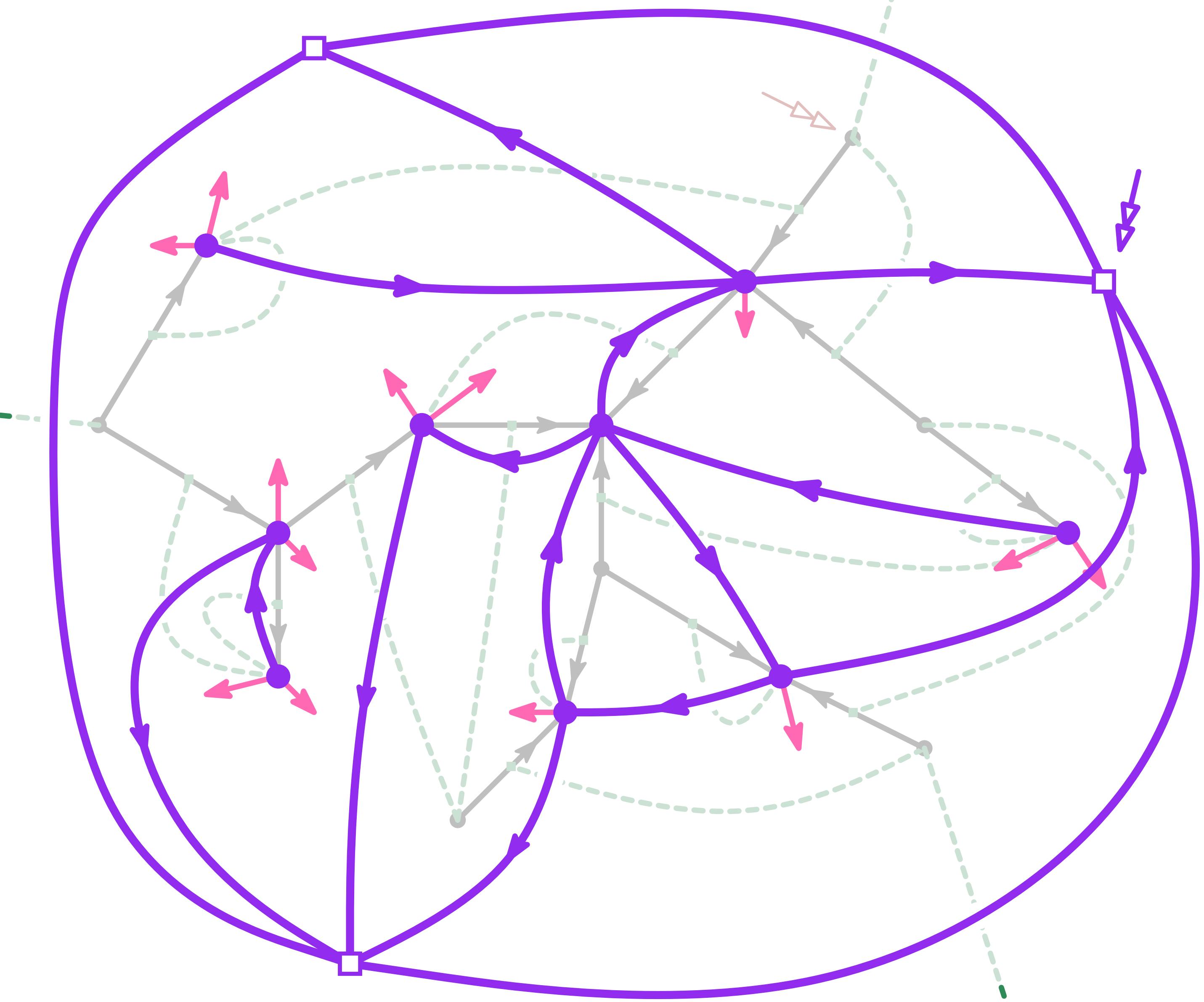

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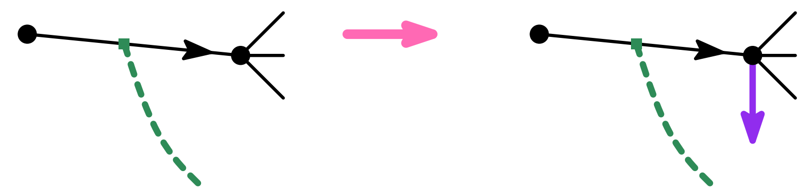

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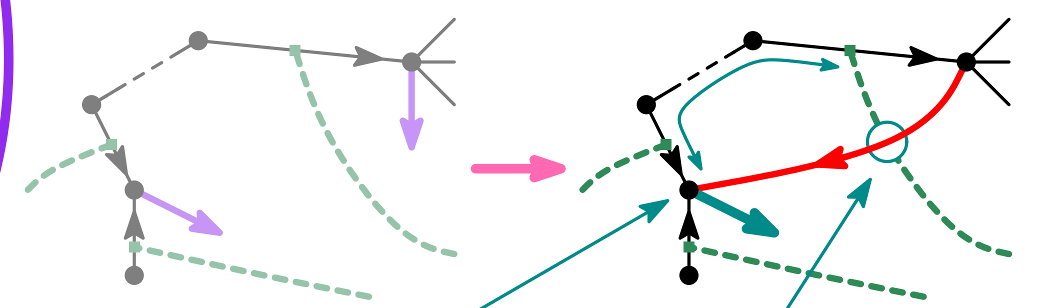

- Add 3 vertices and close the remaining opening stems sector by sector

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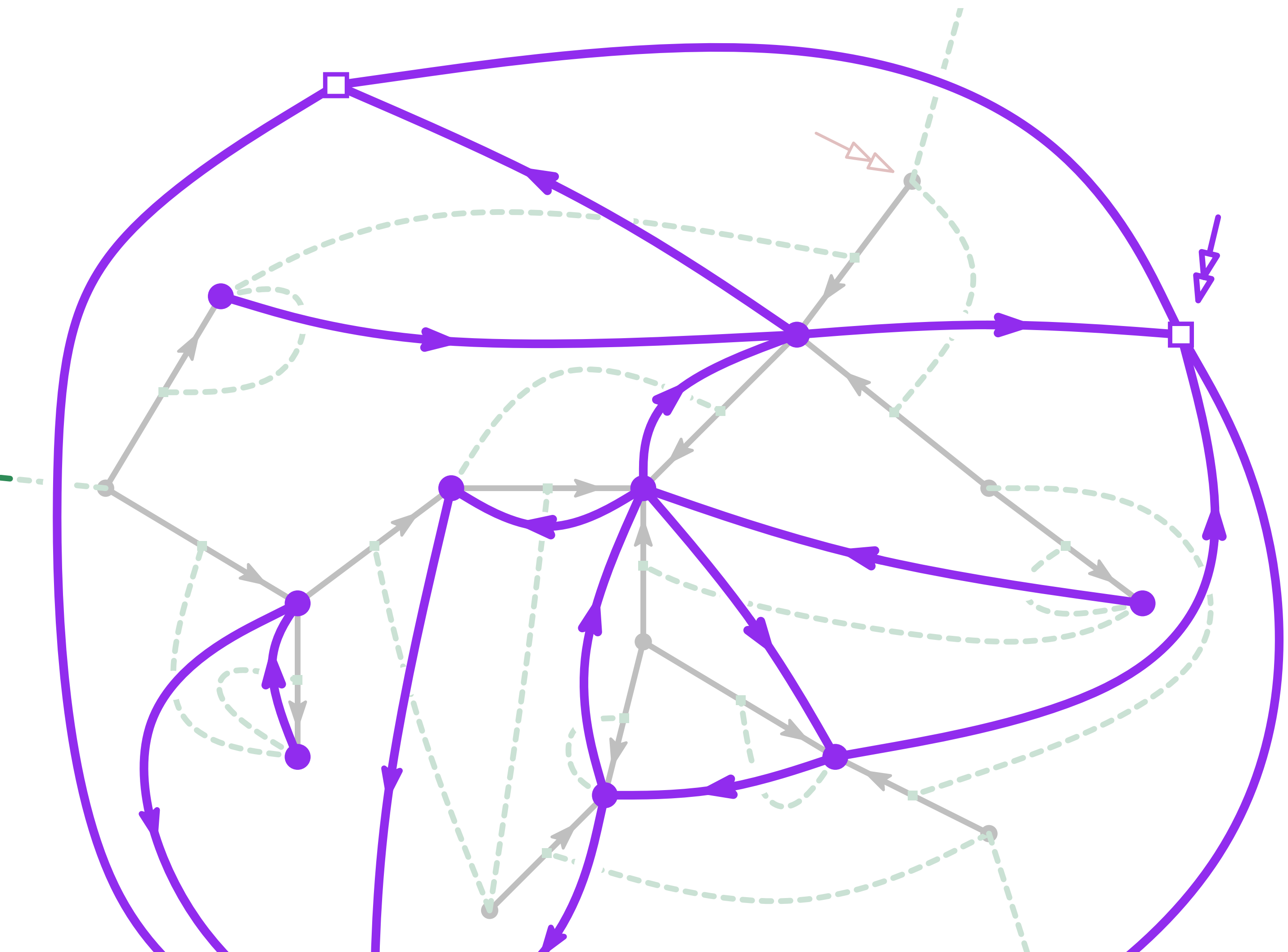
From bipartite cubic maps to simple maps



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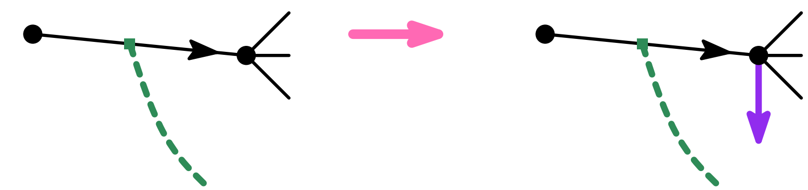

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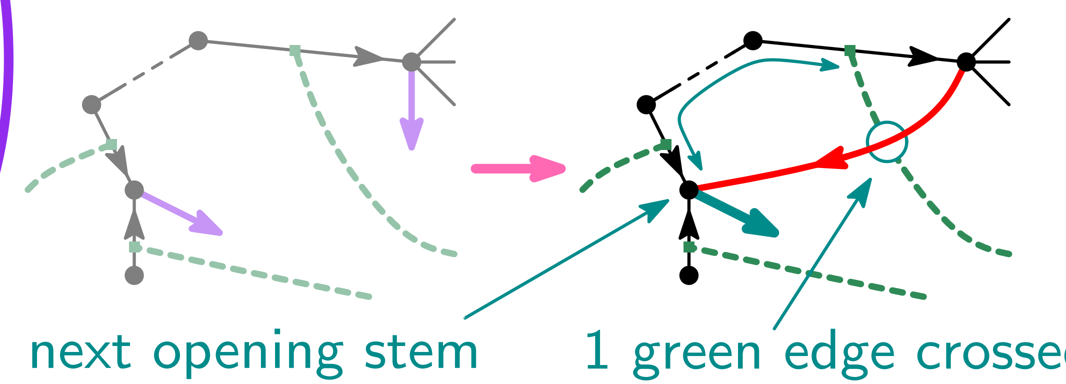
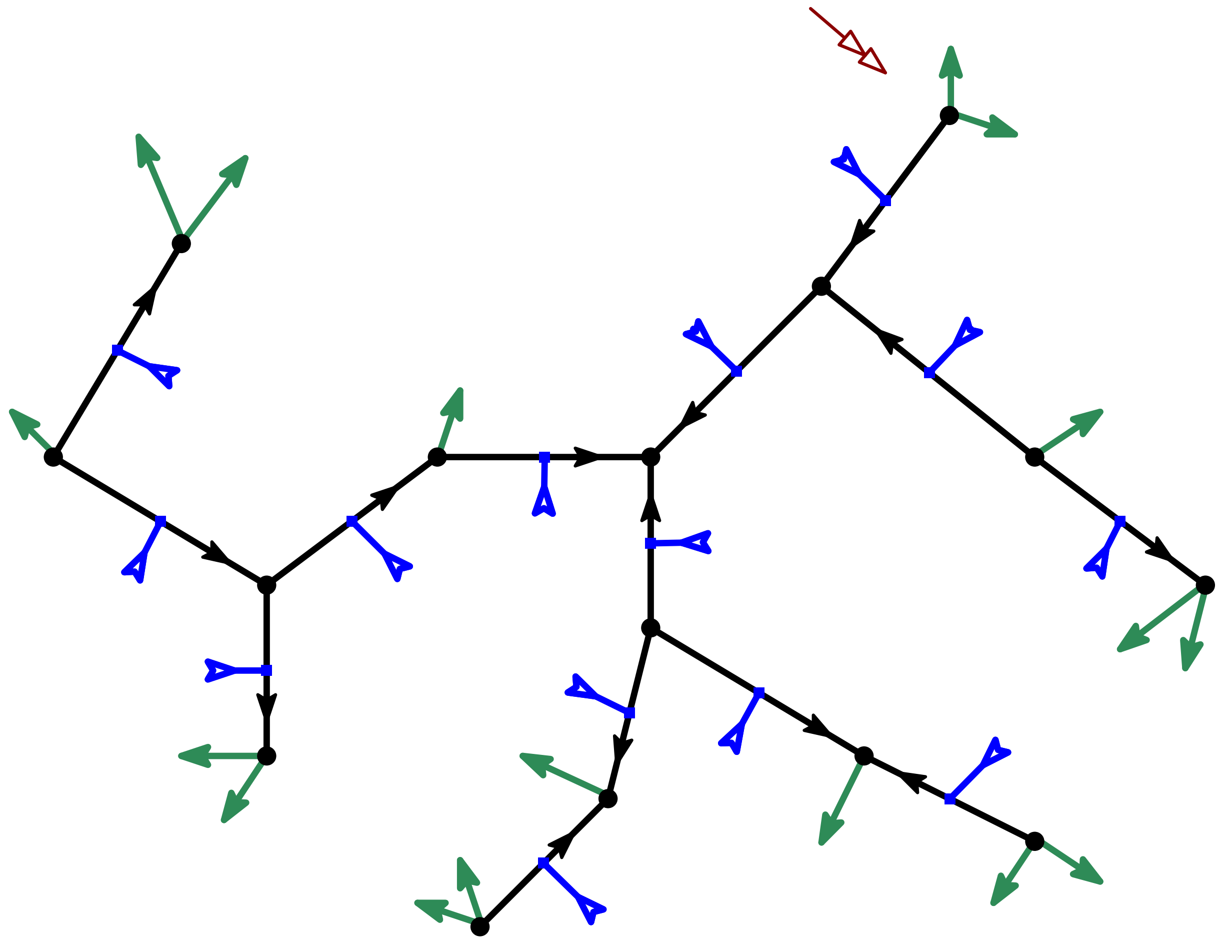


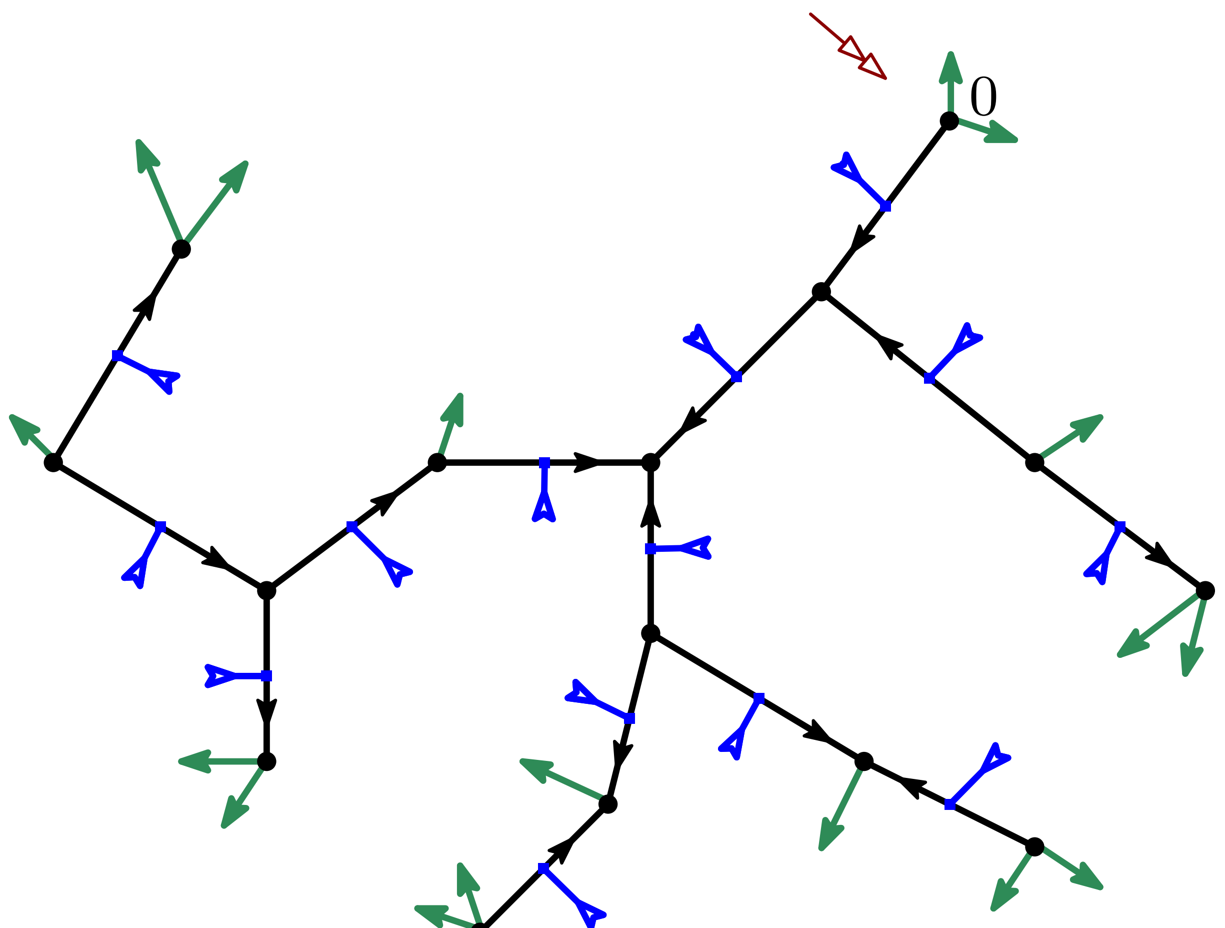
Corollary : [ABCF] We get a bijection between

outer-triangular simple maps and **balanced oriented binary trees**
(with $n+3$ edges) (with n edges)

Same bijection with labels

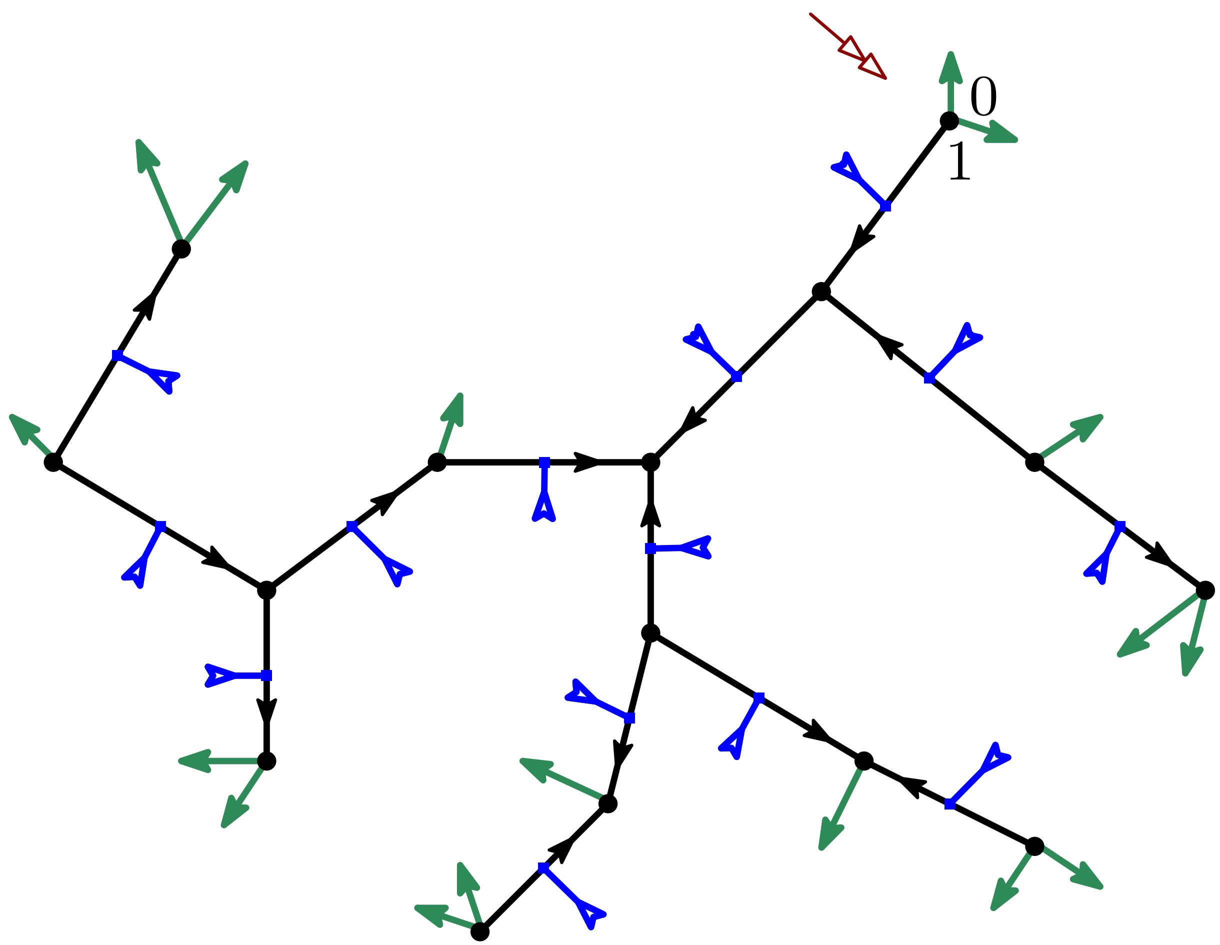


Same bijection with labels



- Label 0 the first corner.

Same bijection with labels

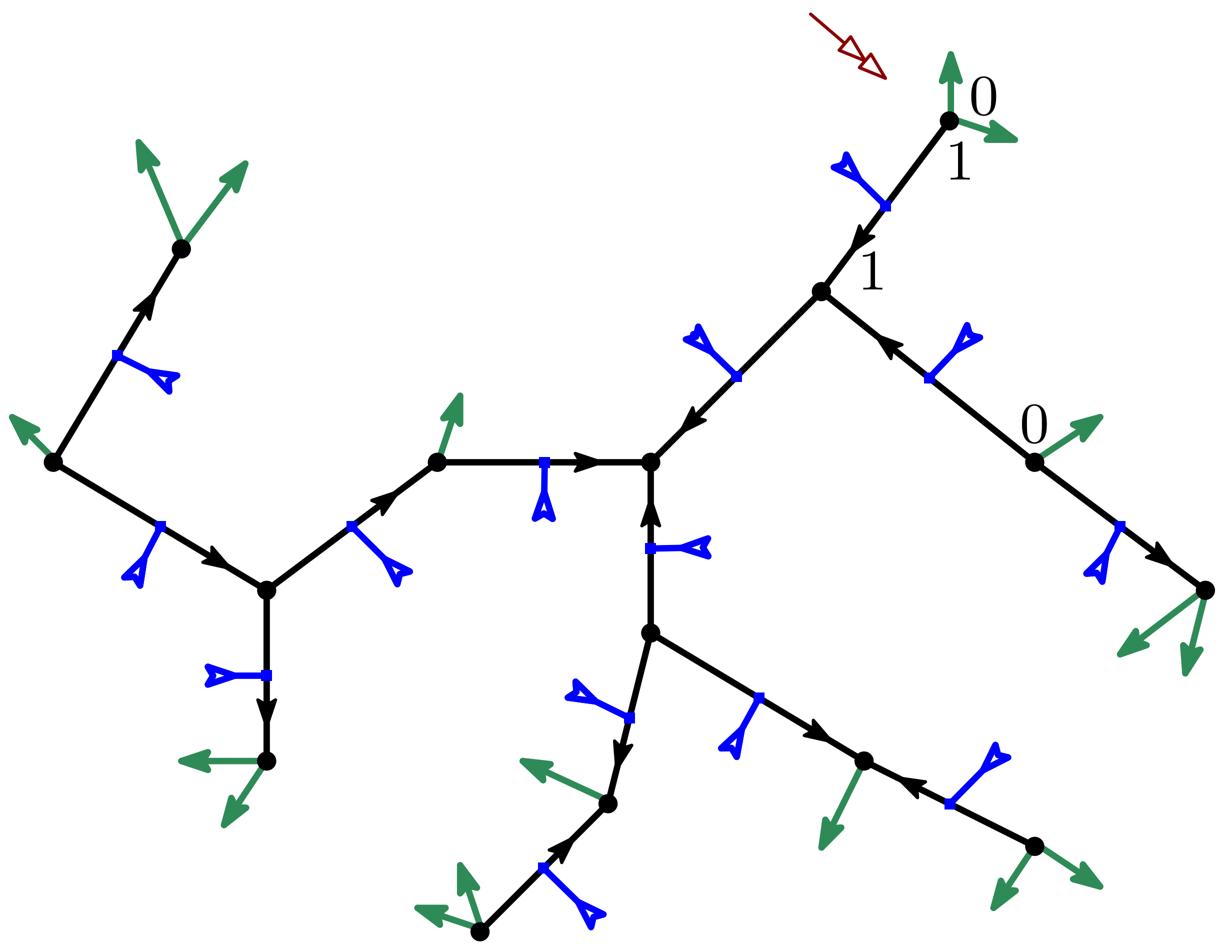


- Label 0 the first corner.

In clockwise order, apply the following rules:

- After an opening stem : increase by 1.

Same bijection with labels

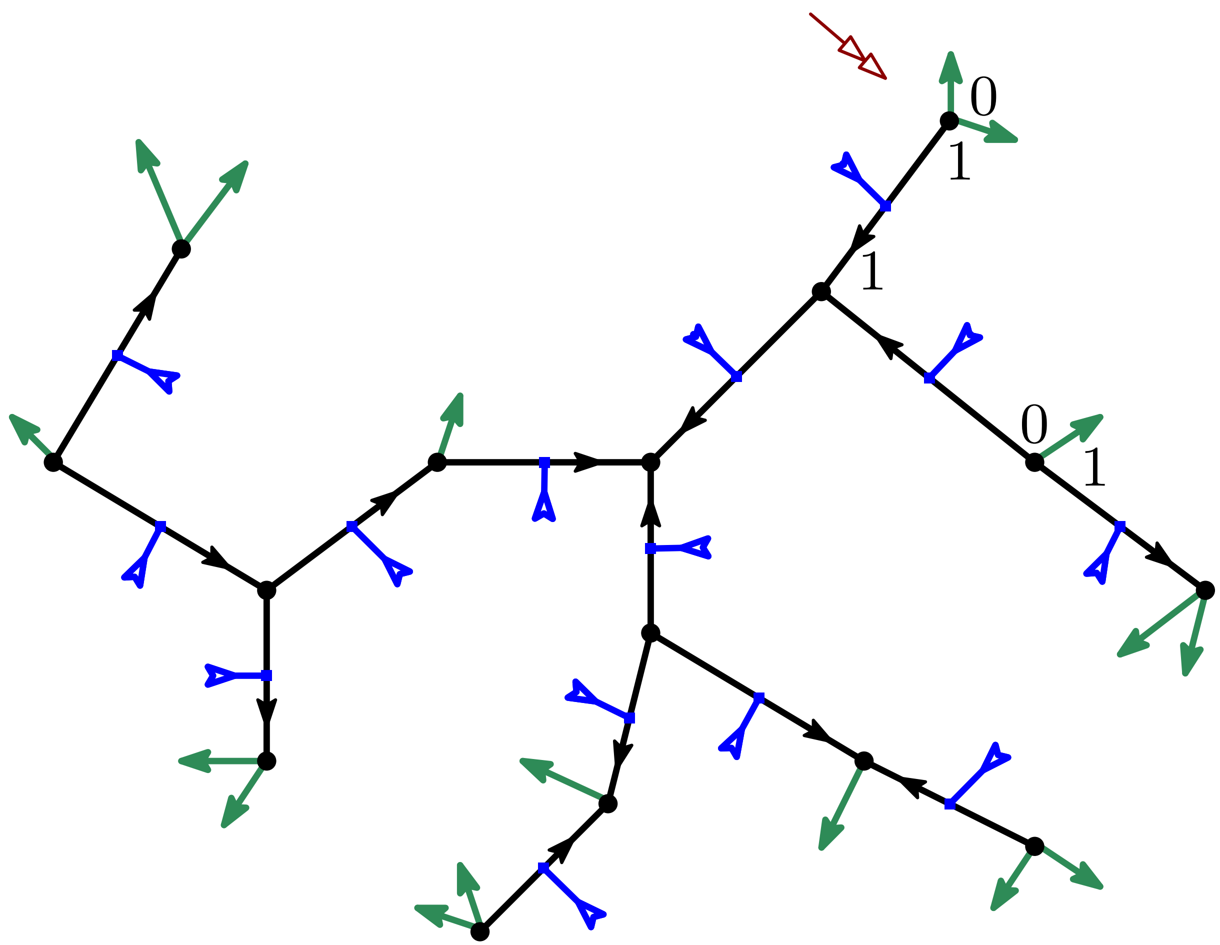


- Label 0 the first corner.

In clockwise order, apply the following rules:

- After an opening stem : increase by 1.
- After a closing stem : decrease by 1.

Same bijection with labels



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- After an opening stem : increase by 1.
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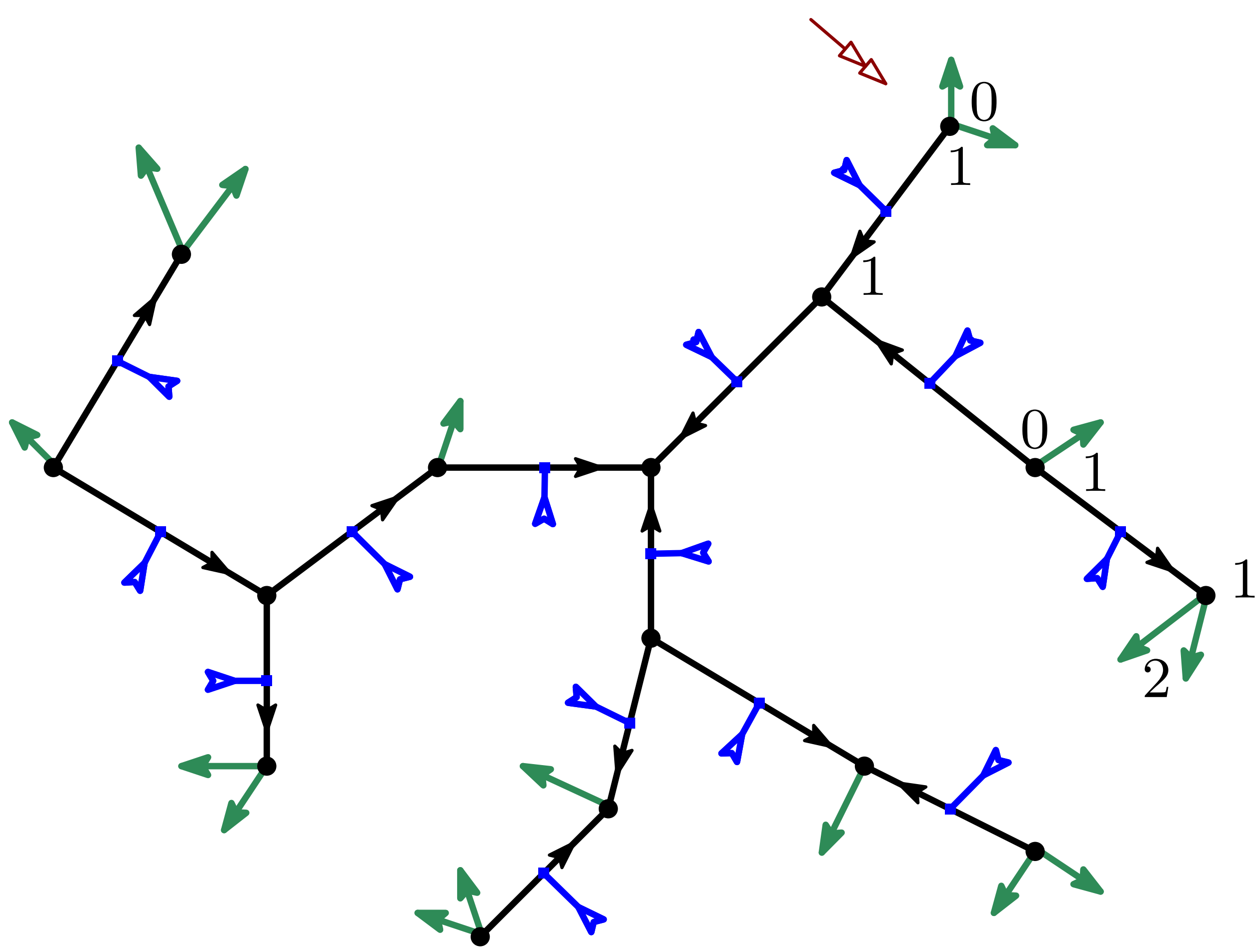
Same bijection with labels



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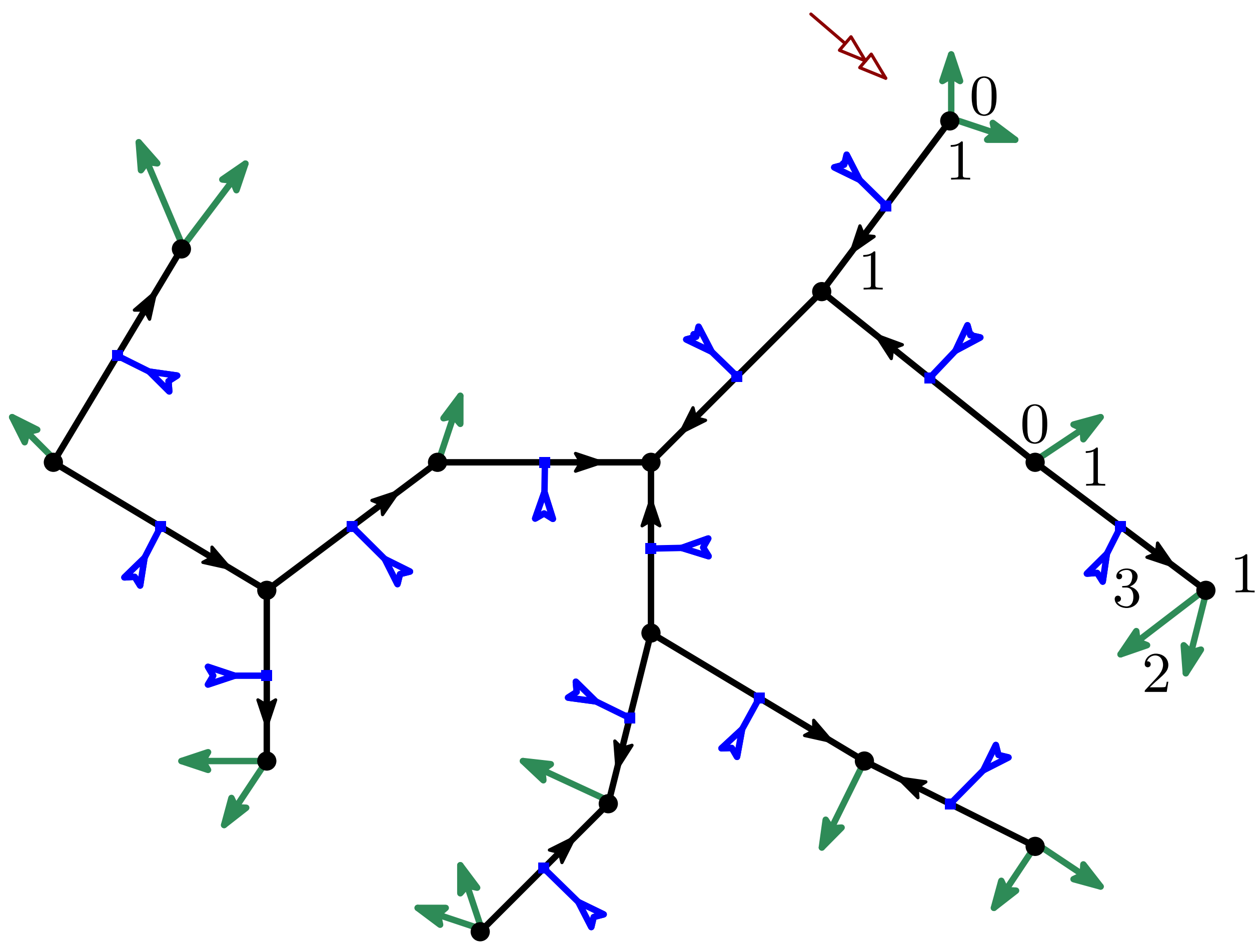
Same bijection with labels



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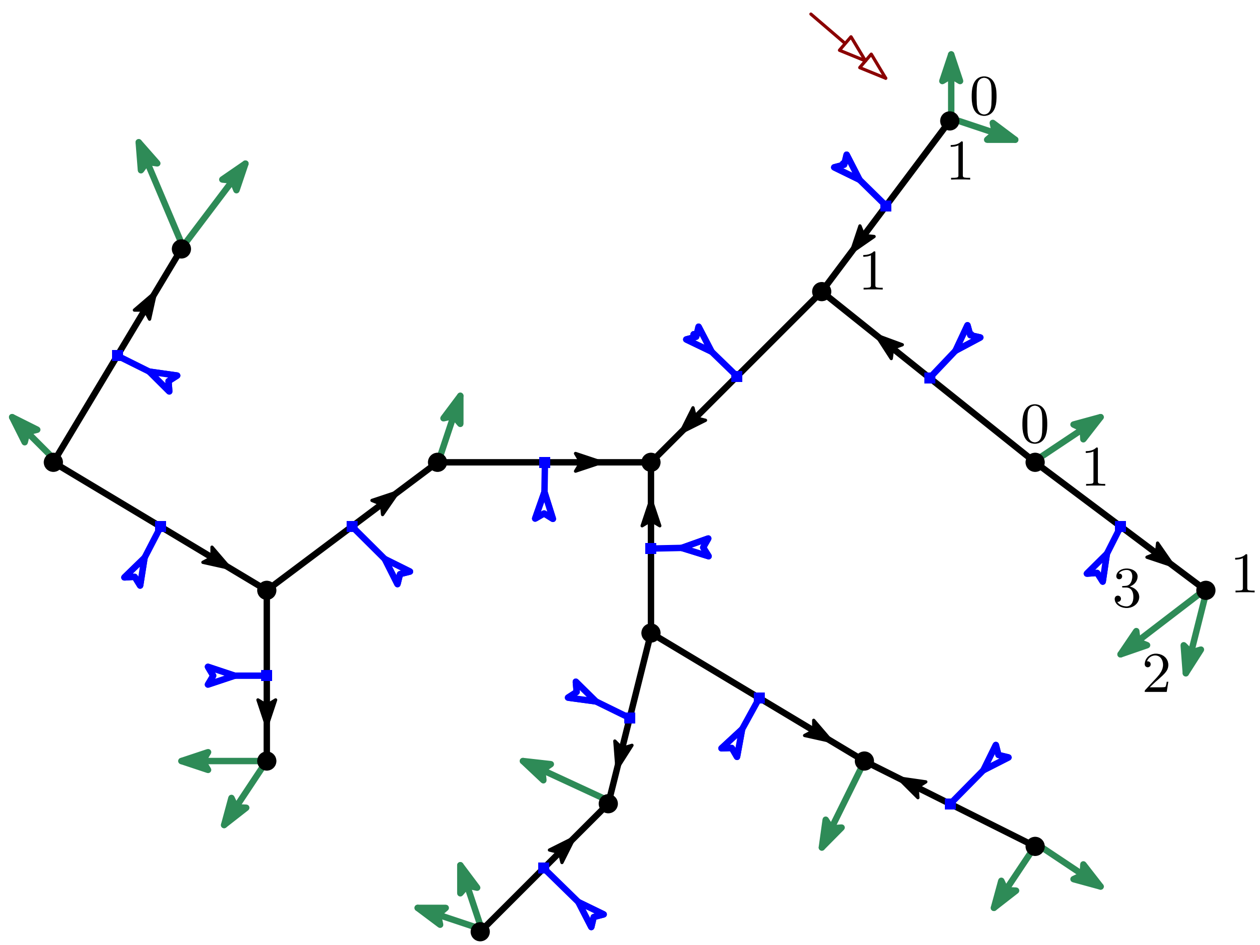
- After a closing stem : decrease by 1.

Same bijection with labels

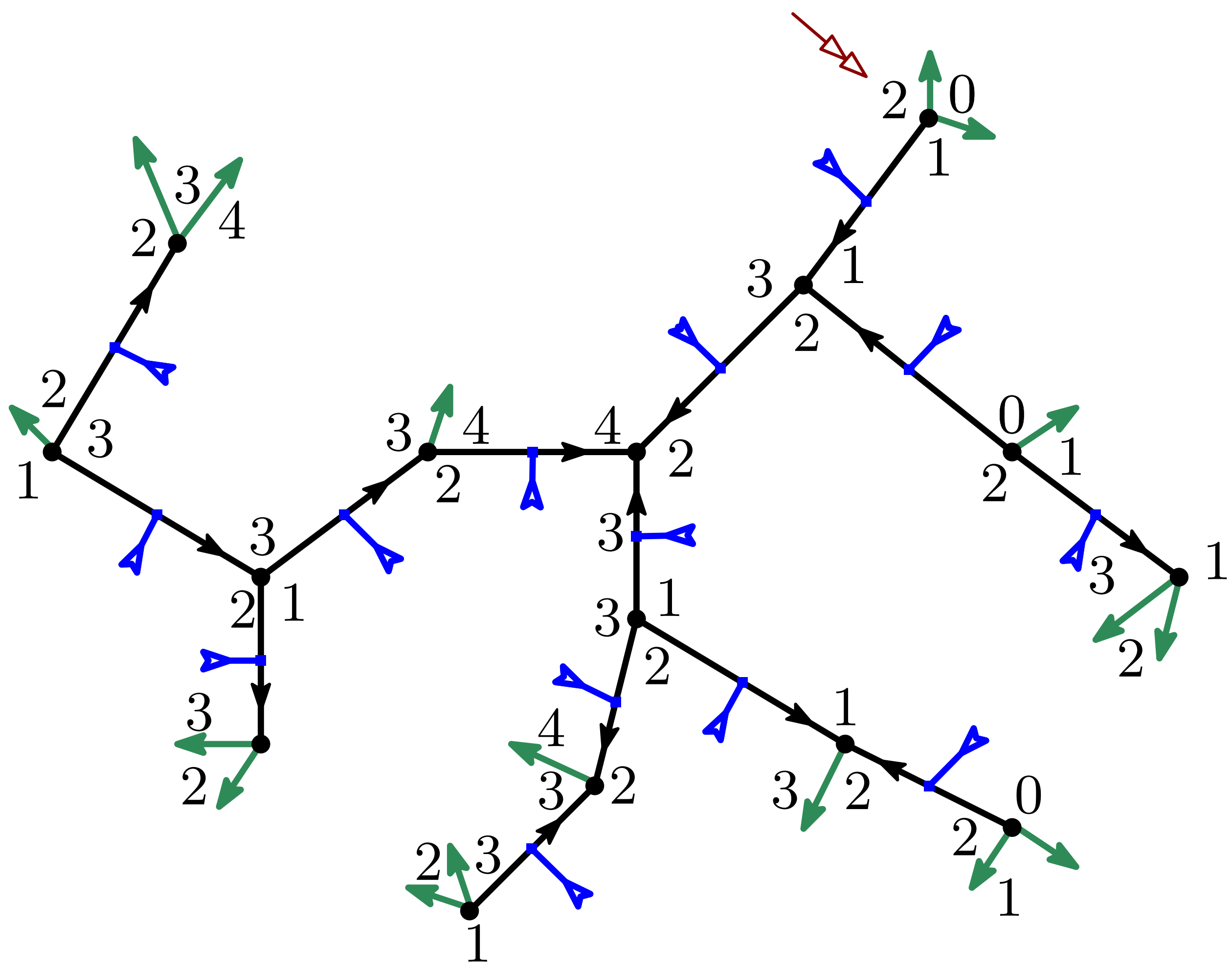


- Label 0 the first corner.

In clockwise order, apply the following rules:

- After an opening stem : increase by 1.
- After a closing stem : decrease by 1.

Same bijection with labels



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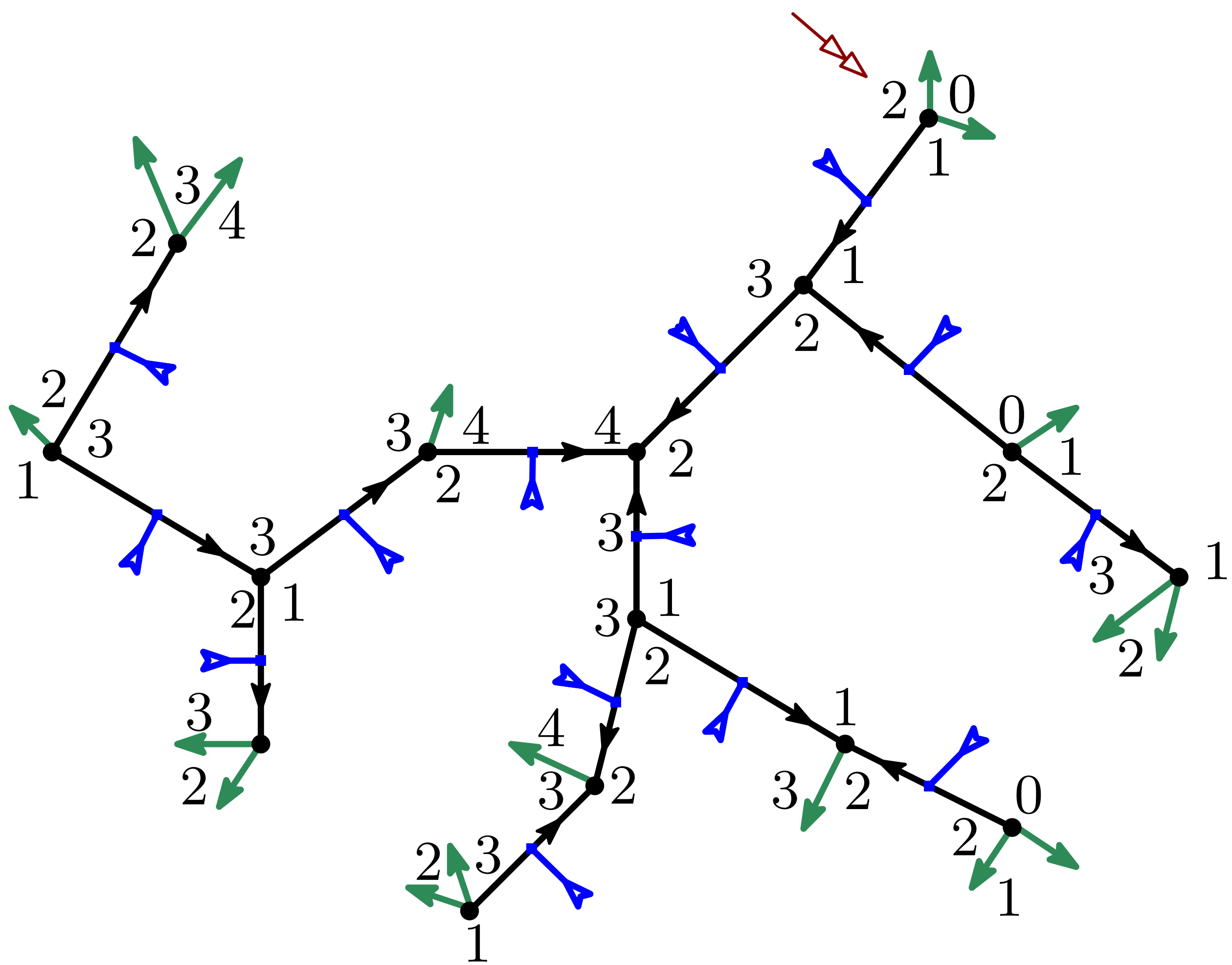

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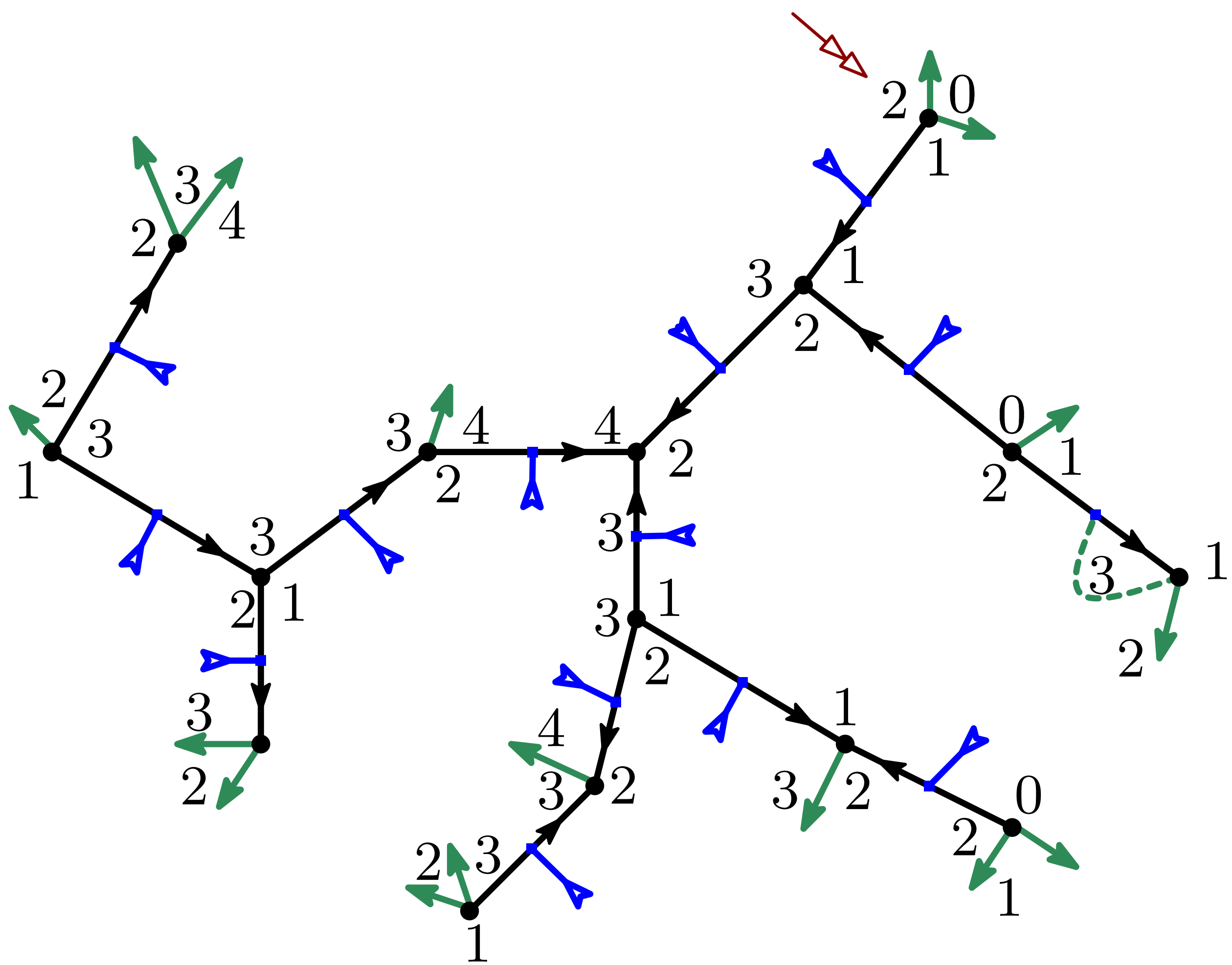
Labels $\approx$ depth of the face in the cubic map

Same bijection with labels



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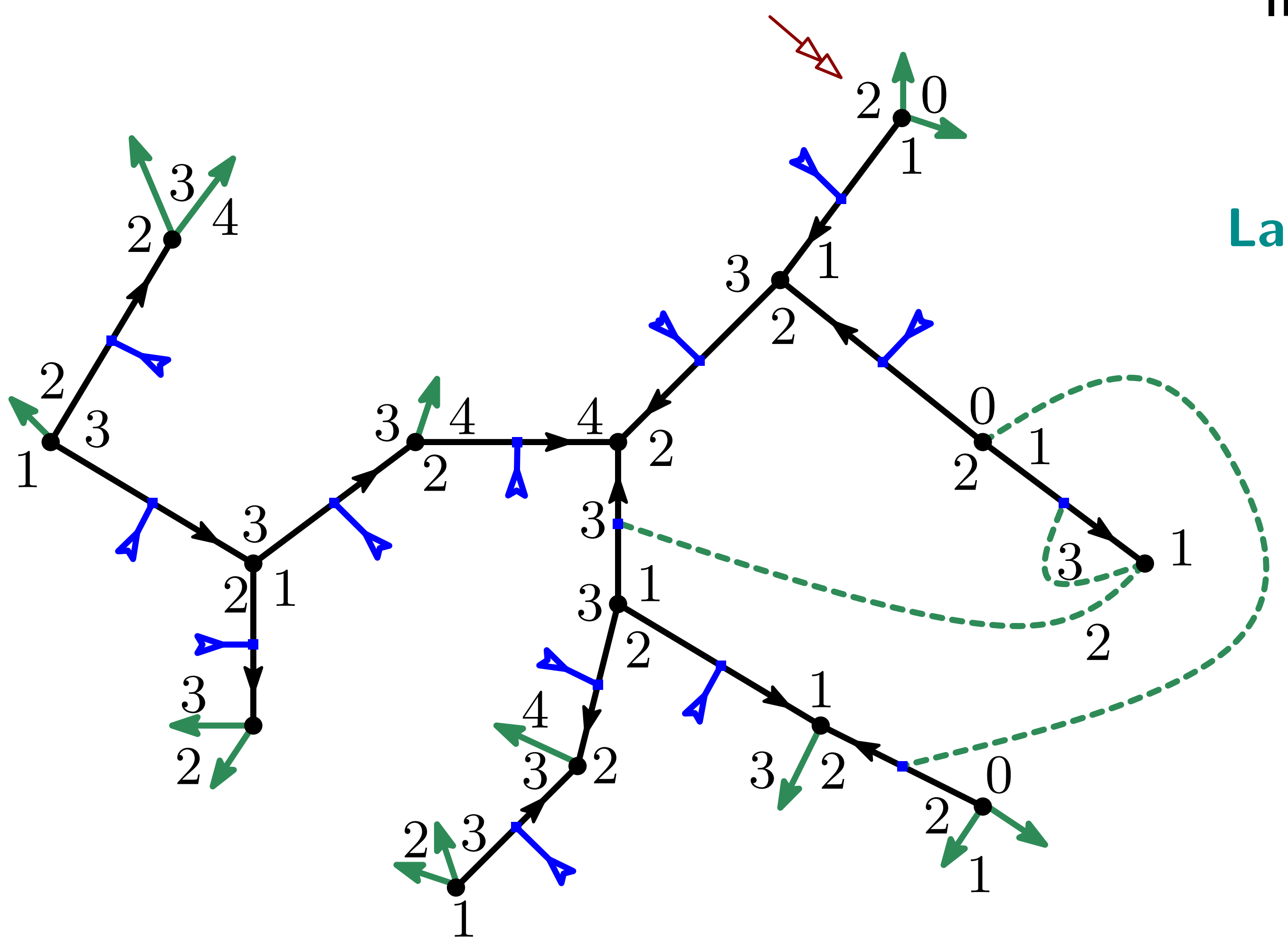

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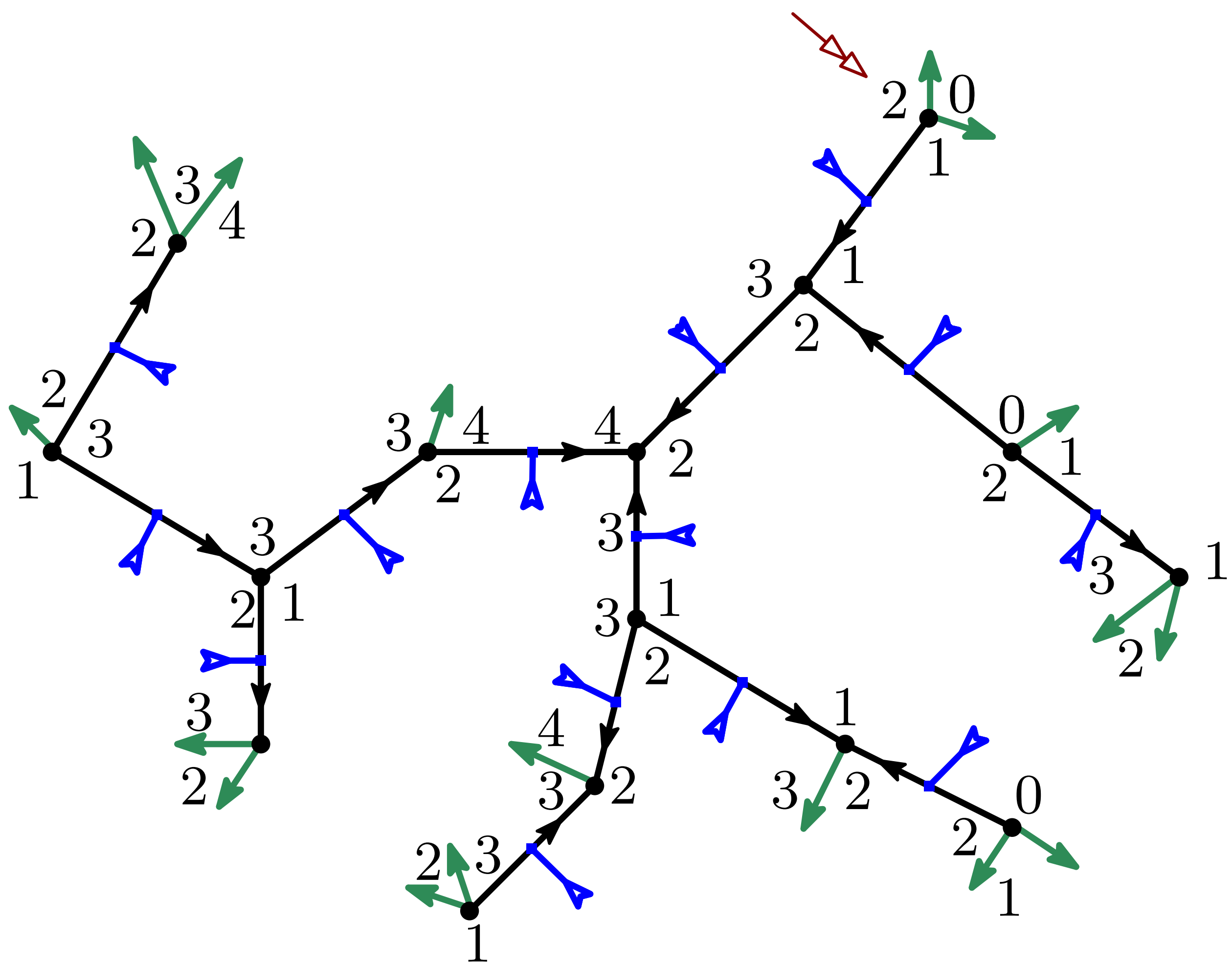
Labels ≈ depth of the face in the cubic map

Same bijection with labels



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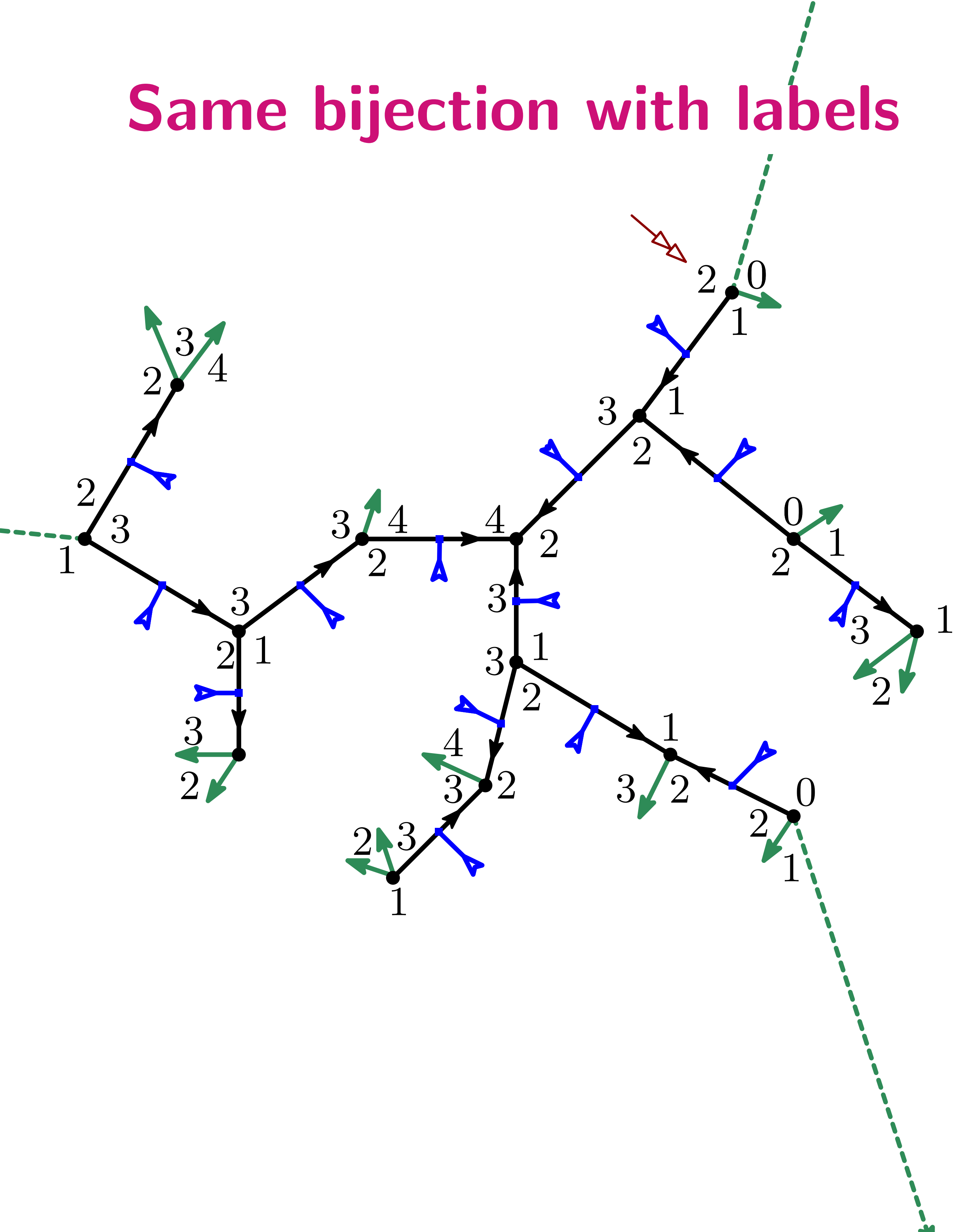

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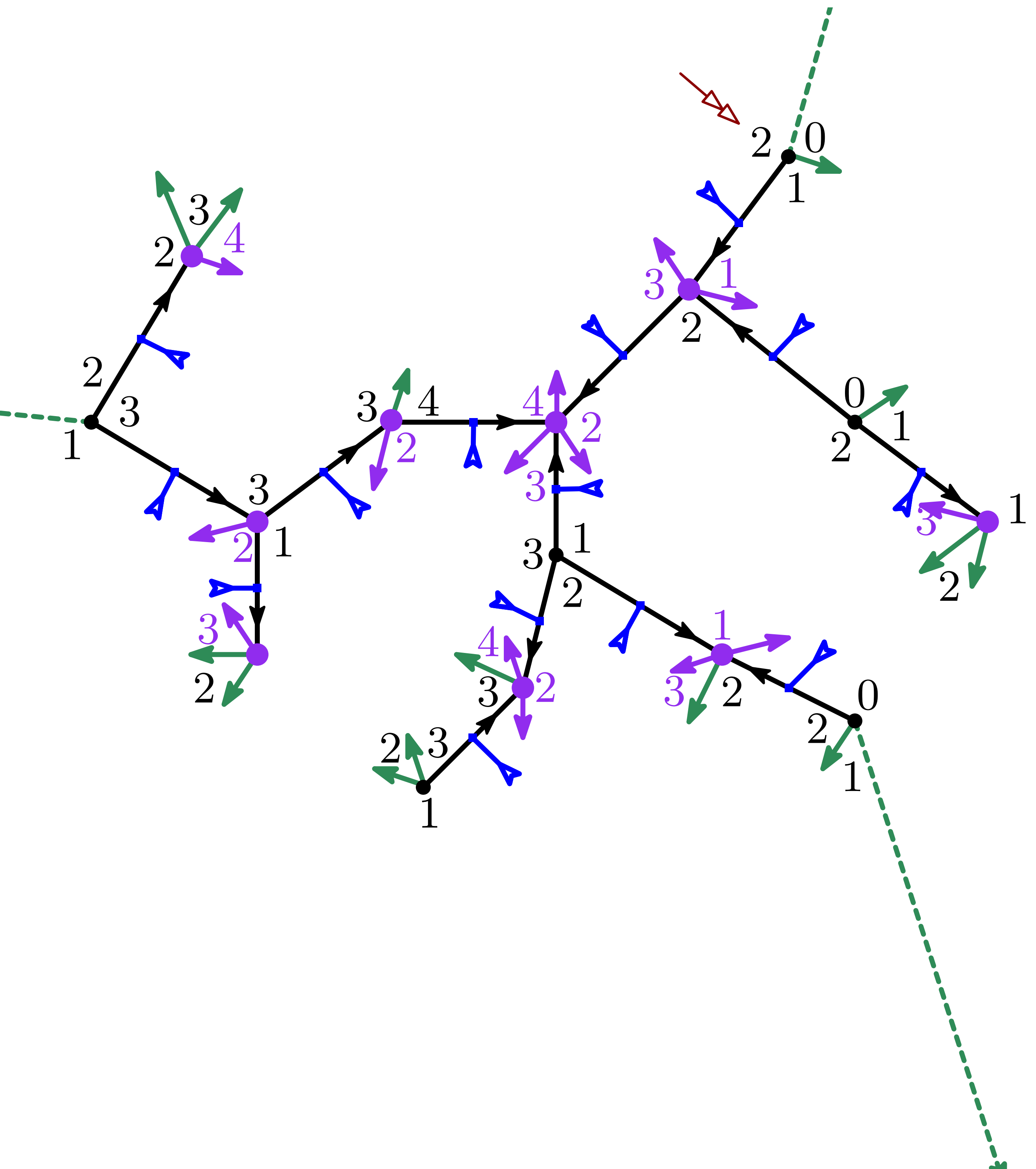
- After a closing stem : decrease by 1.

Labels $\approx$ depth of the face in the cubic map

Unmatched stems = last 0, 1 and 2 corners

Same bijection with labels



- Label 0 the first corner.

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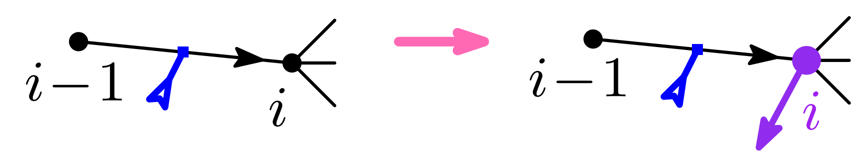
Labels ≈ depth of the face in the cubic map

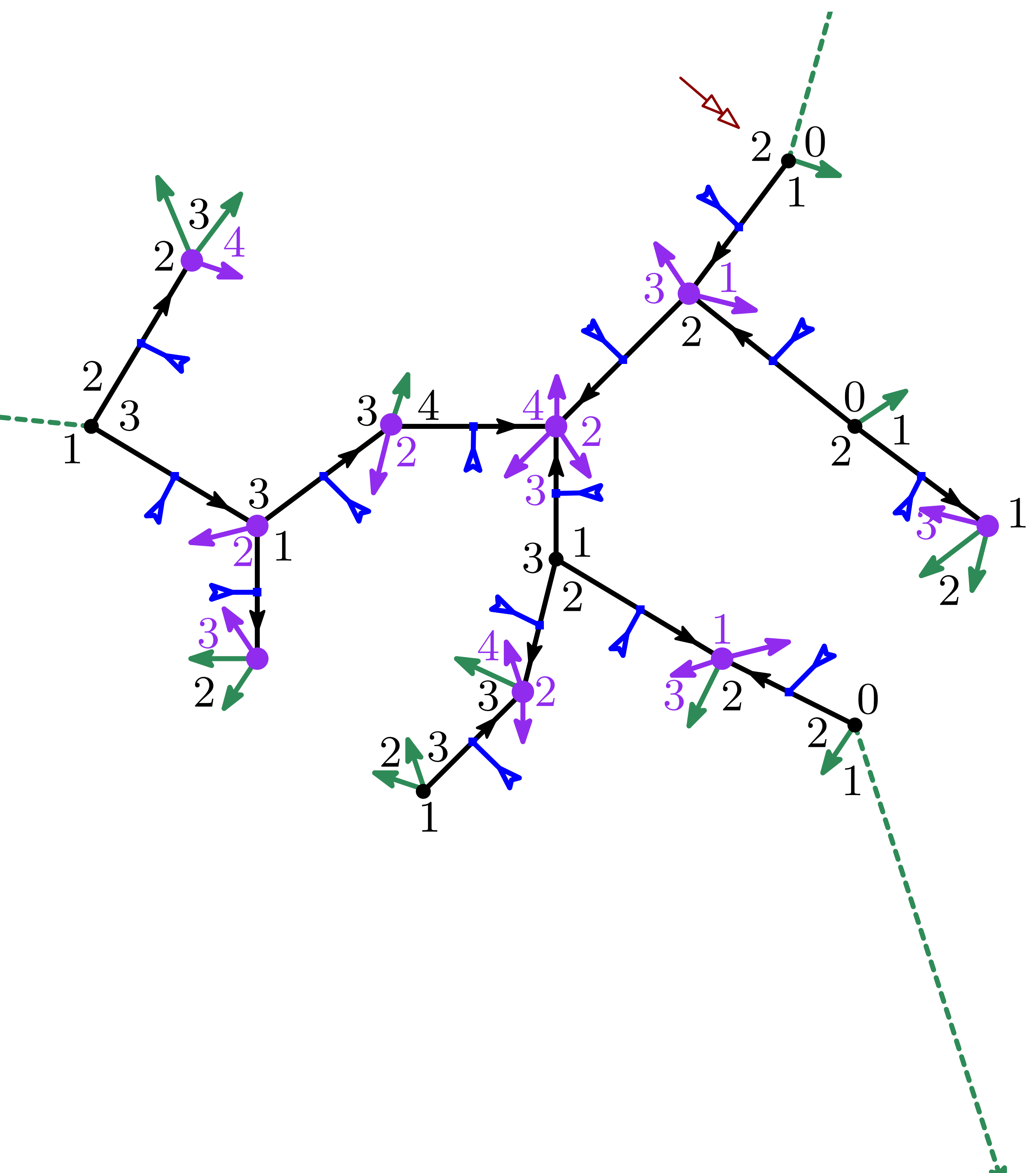
Unmatched stems = last 0, 1 and 2 corners

- Apply the following local rule :



(= add a → before each descent and color the corresponding corner and vertex)

Same bijection with labels



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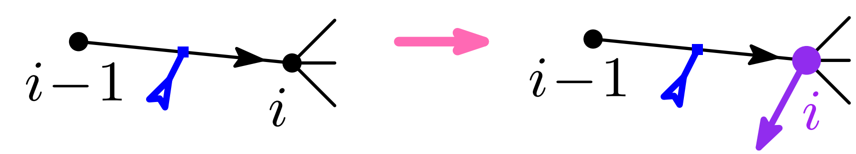
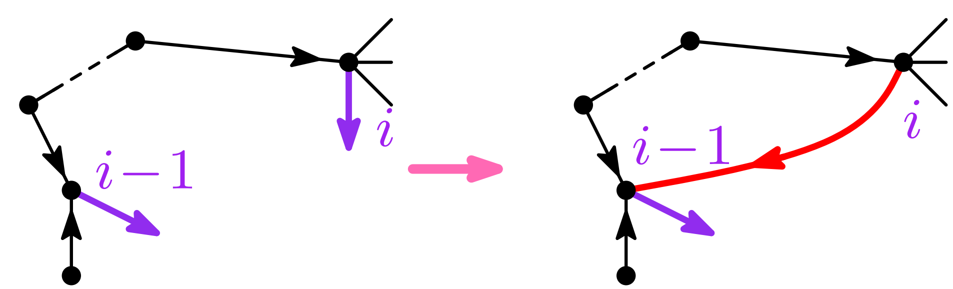


(= add a → before each descent and color the corresponding corner and vertex)

- Erase all non-purple and do the following closures



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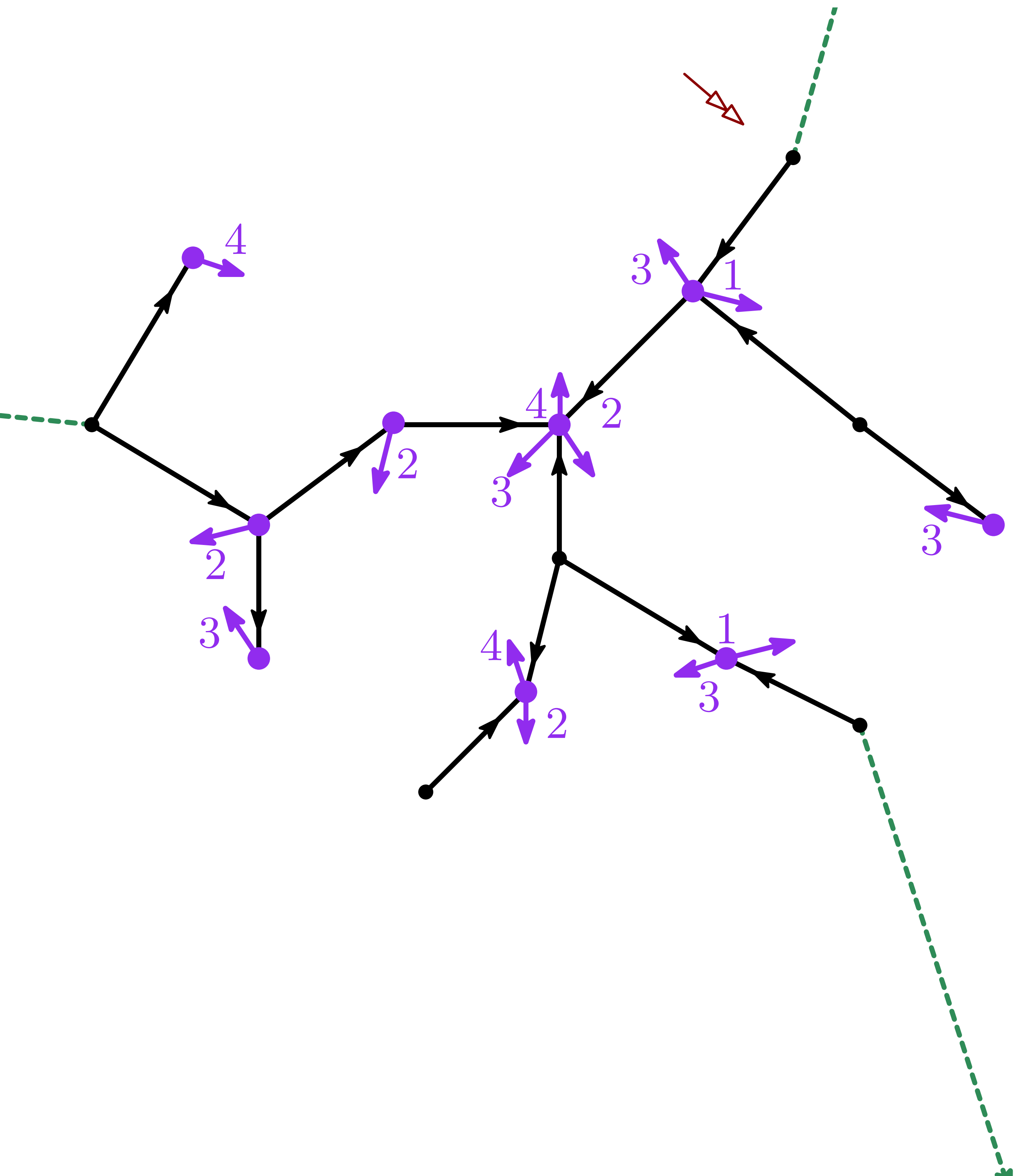

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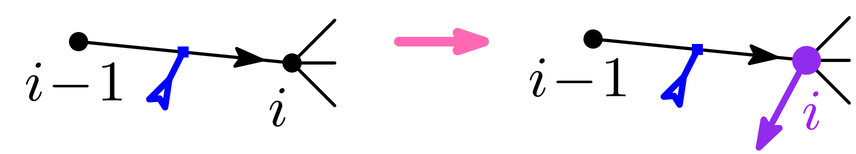

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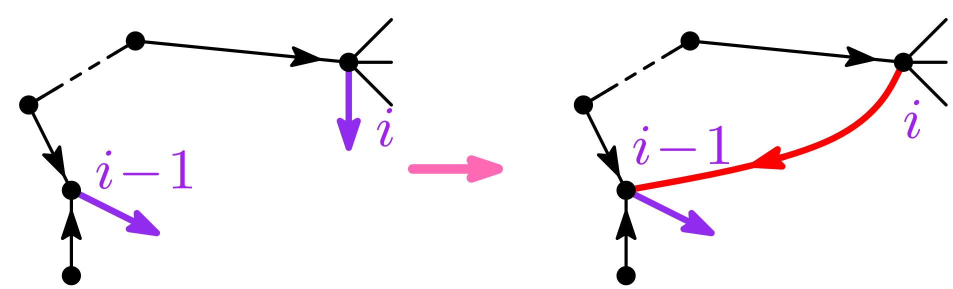

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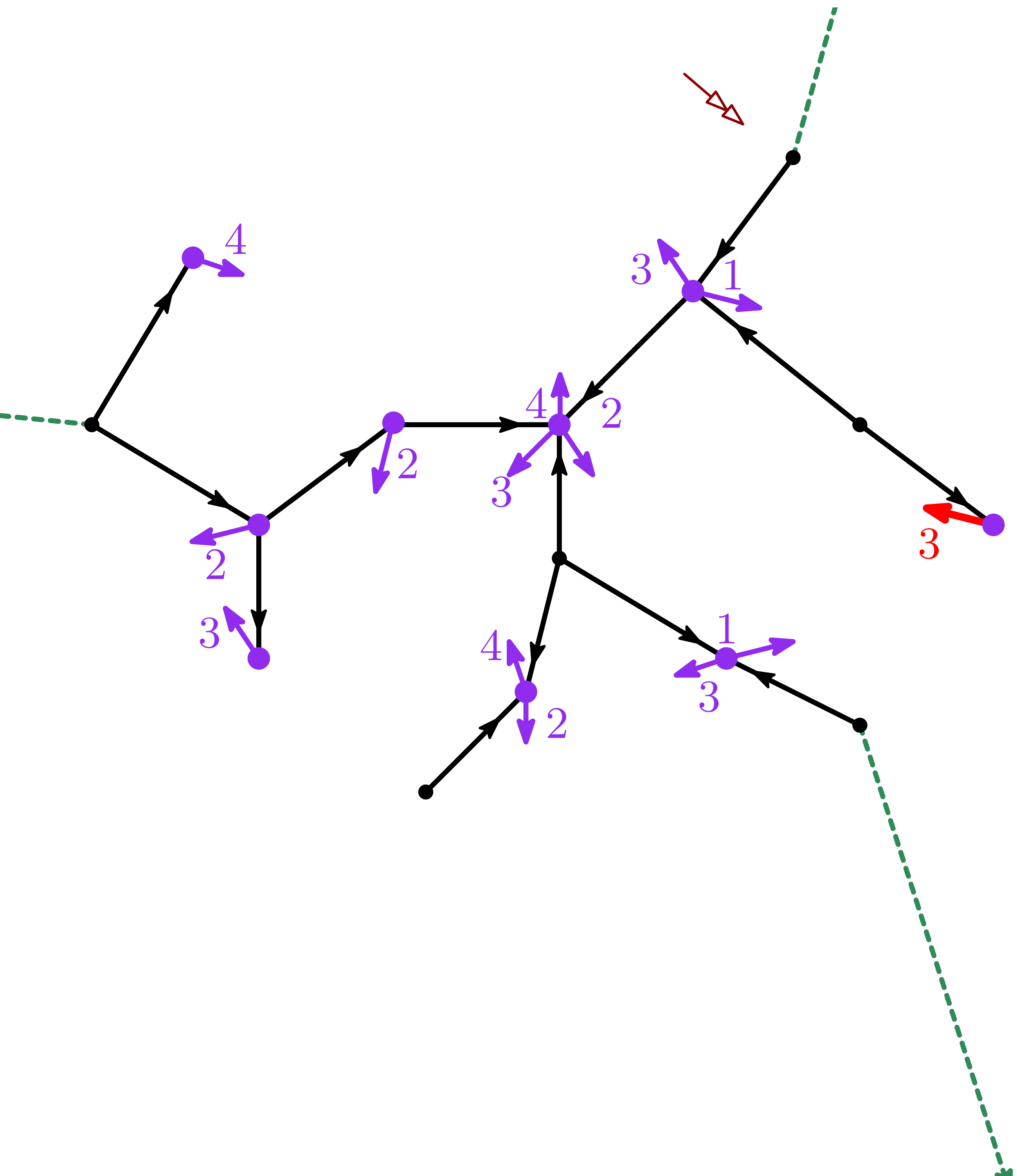

- Label 0 the first corner.

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Labels $\approx$ depth of the face in the cubic map

Unmatched stems = last 0, 1 and 2 corners

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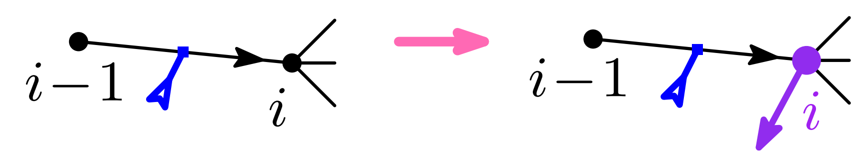

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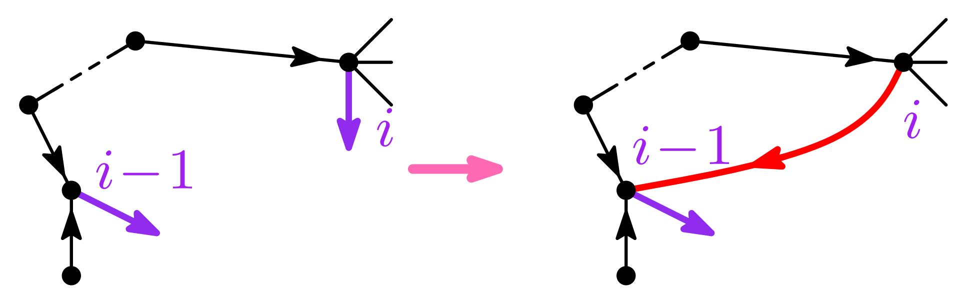

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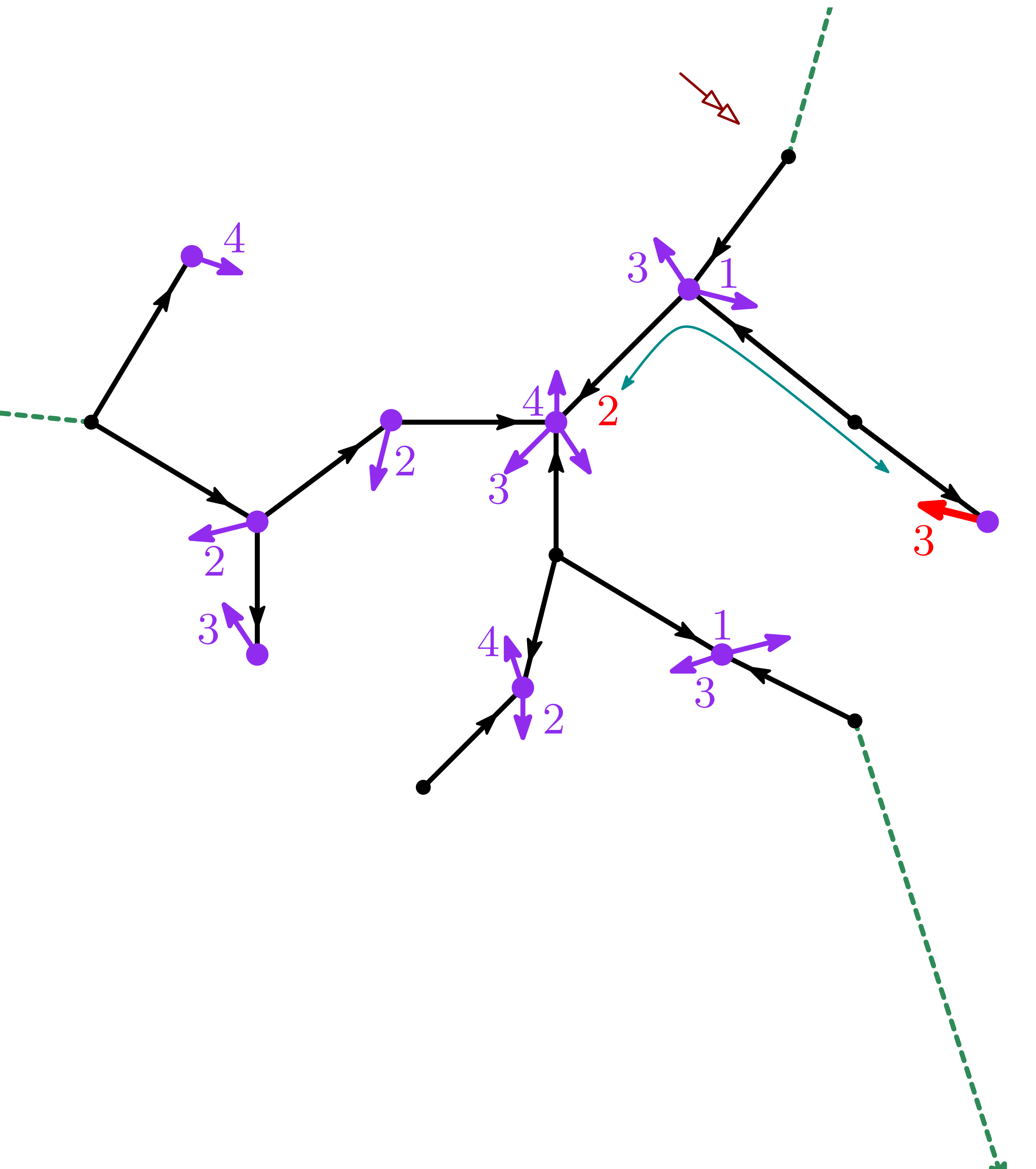

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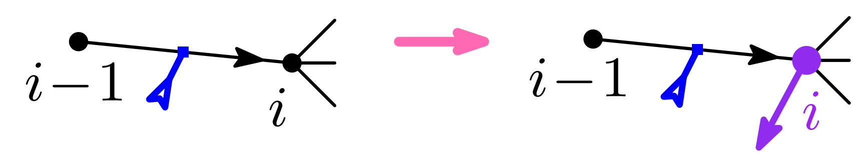

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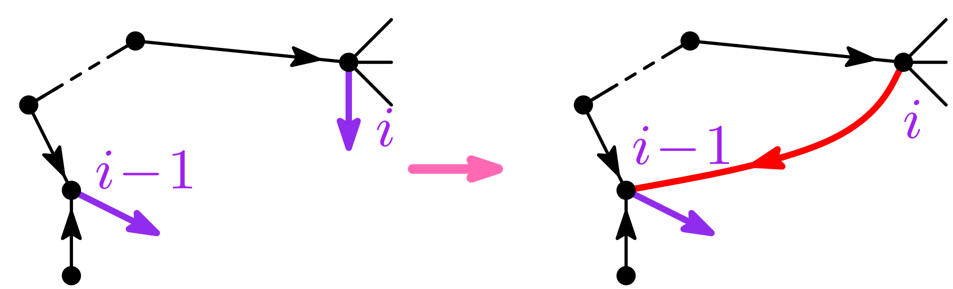

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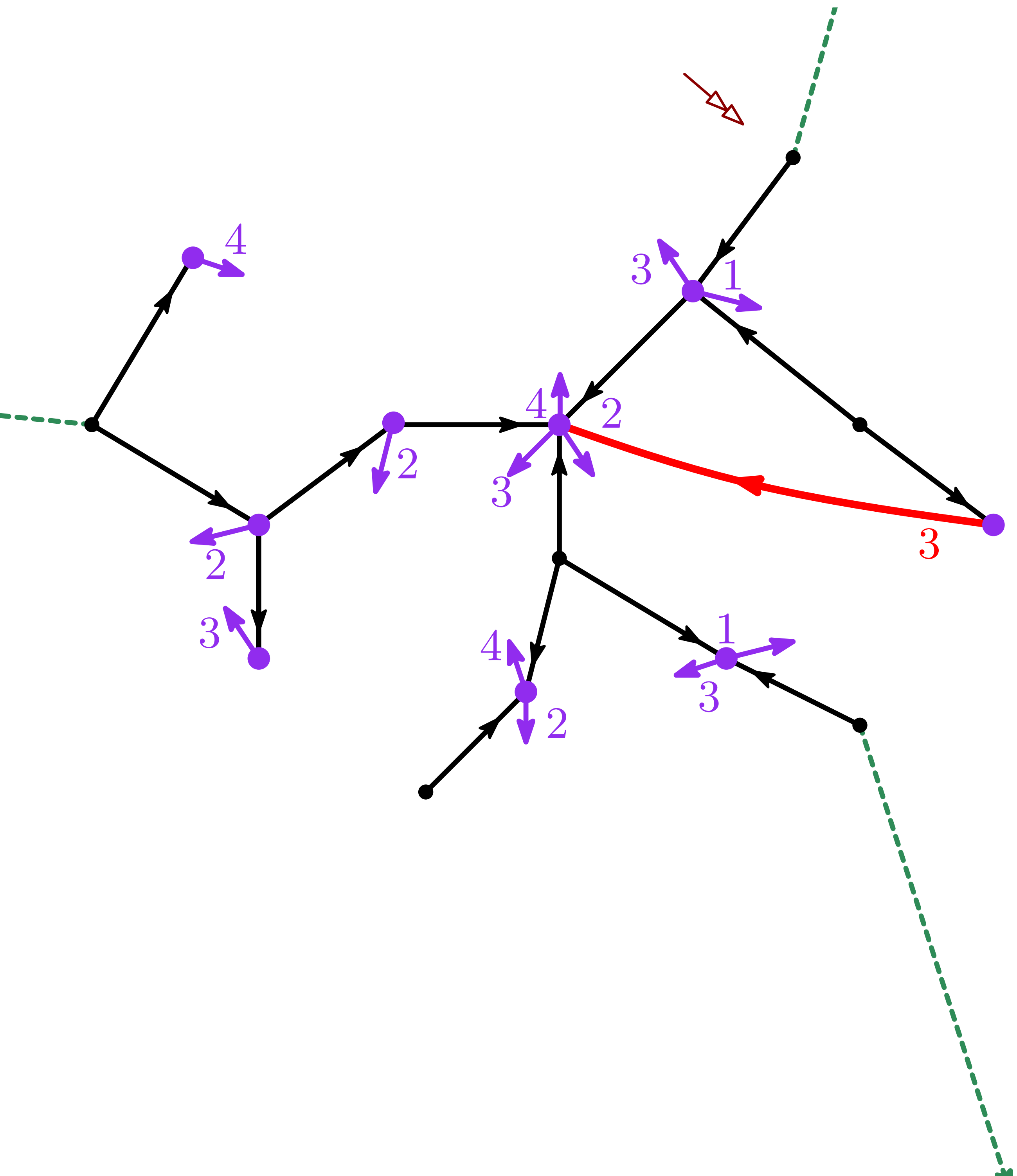

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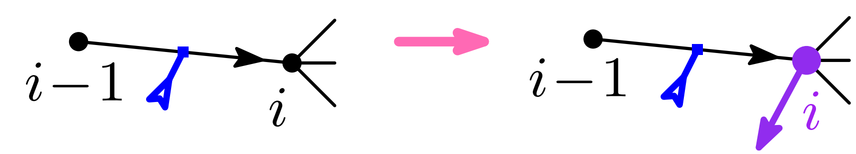

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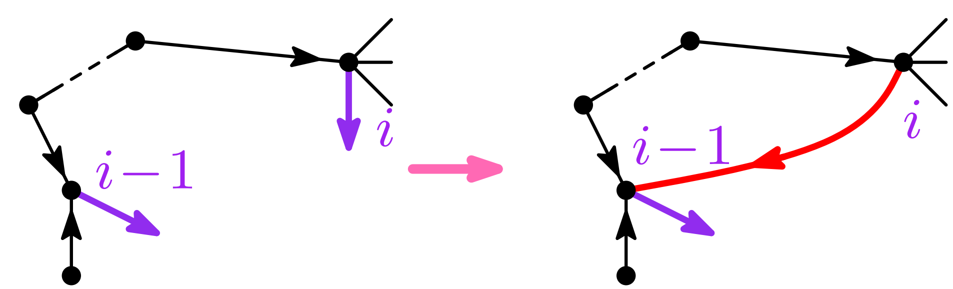

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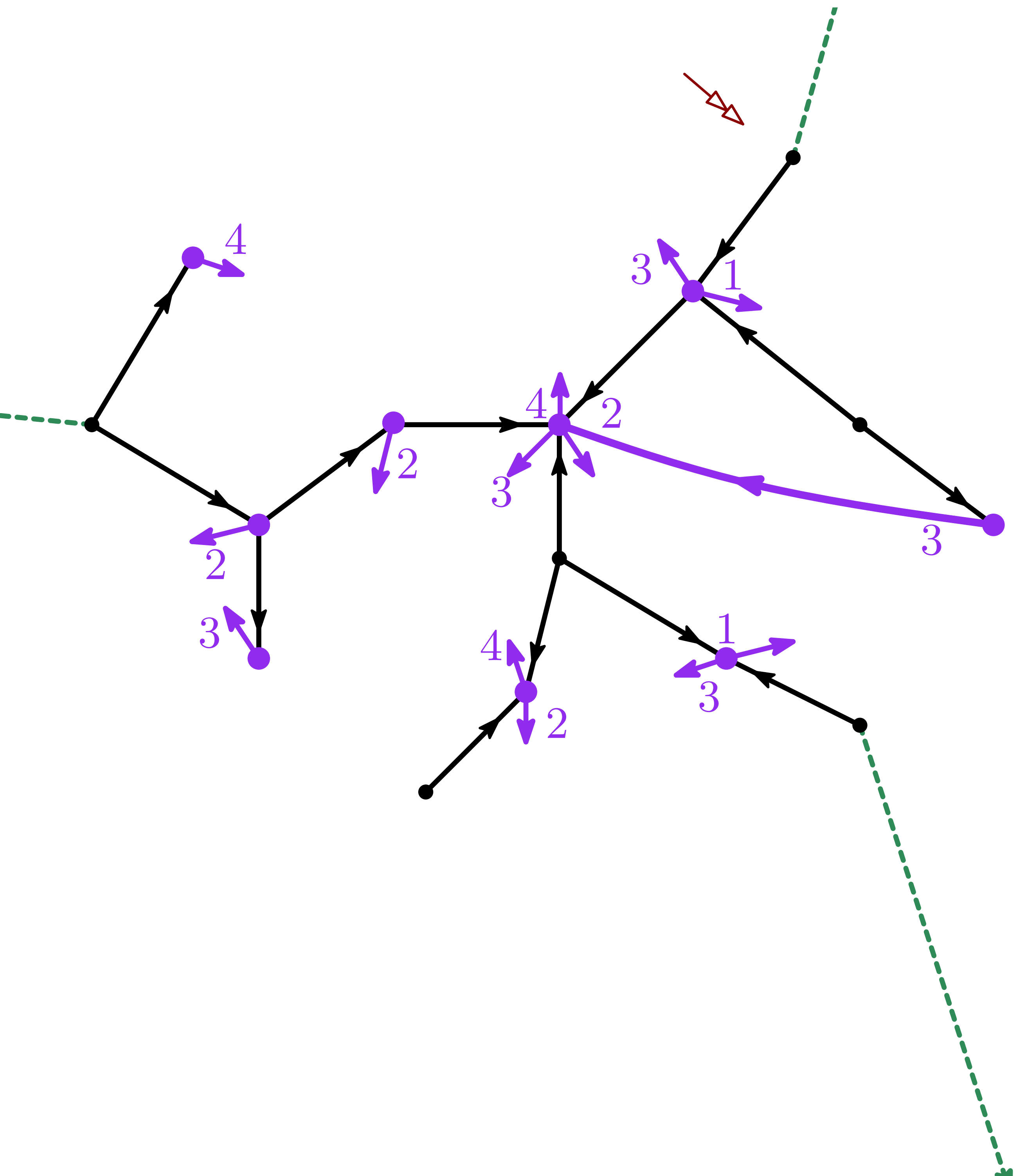

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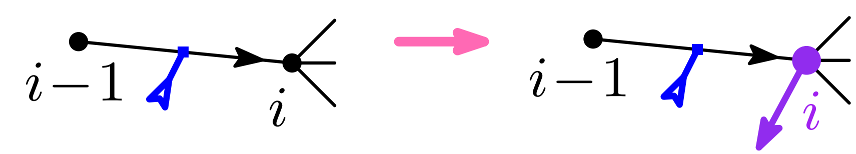

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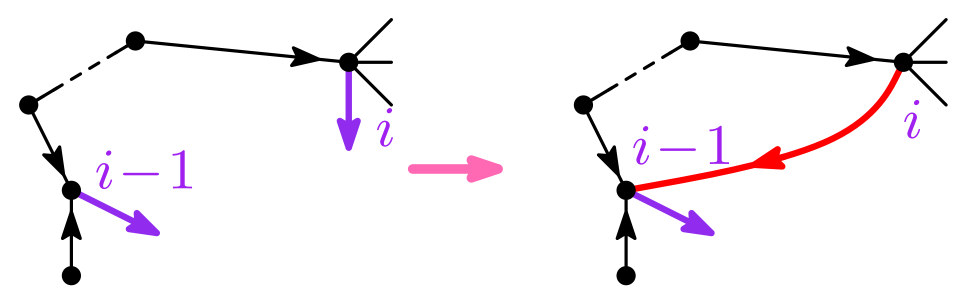

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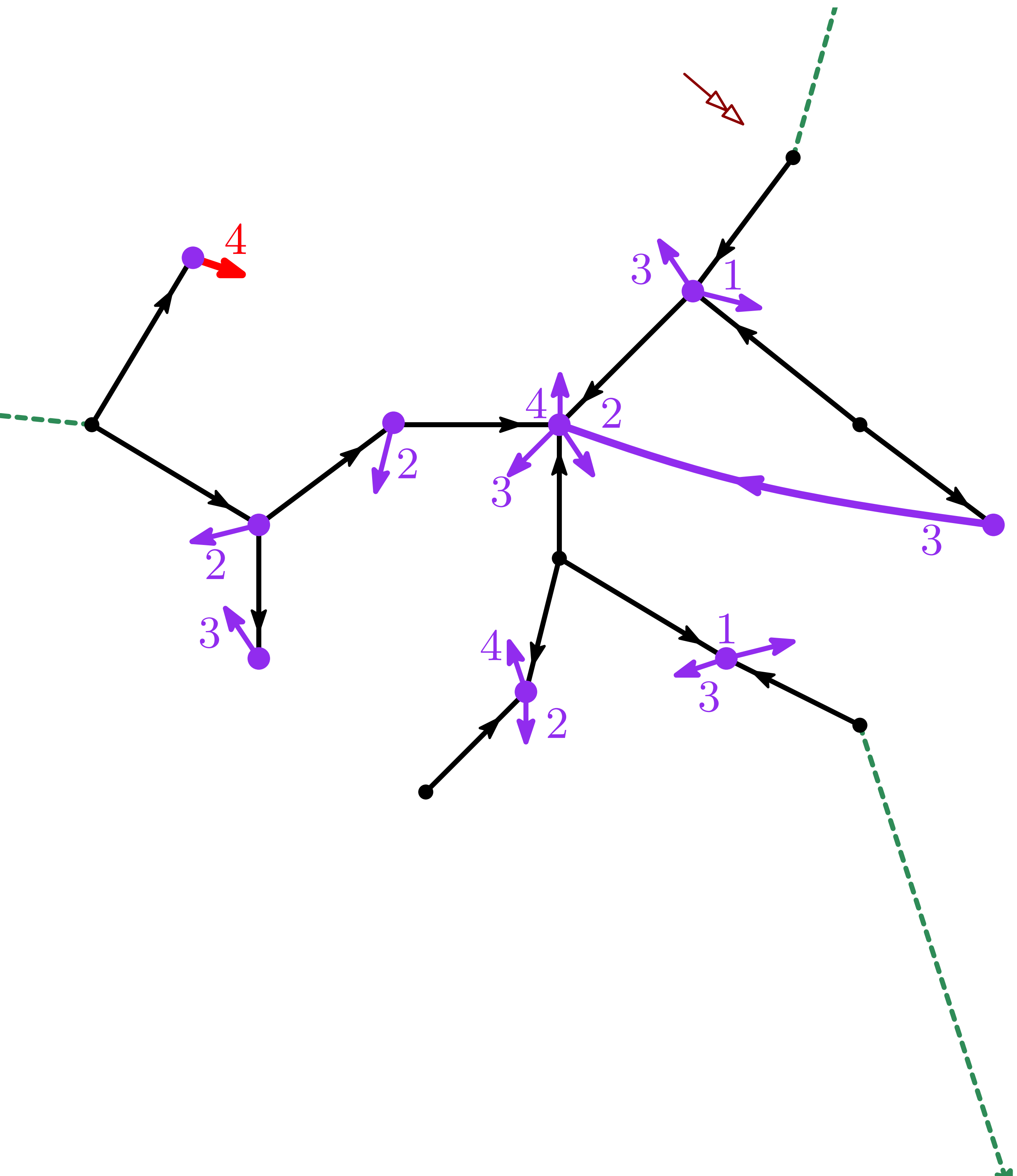

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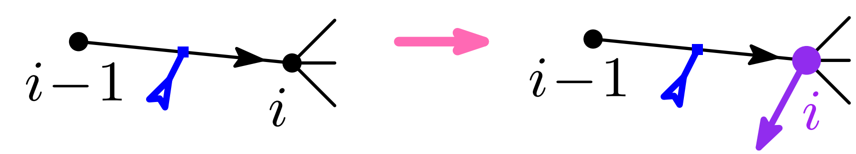

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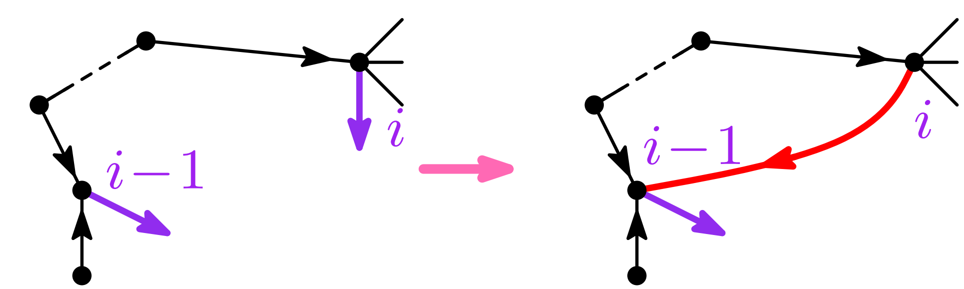

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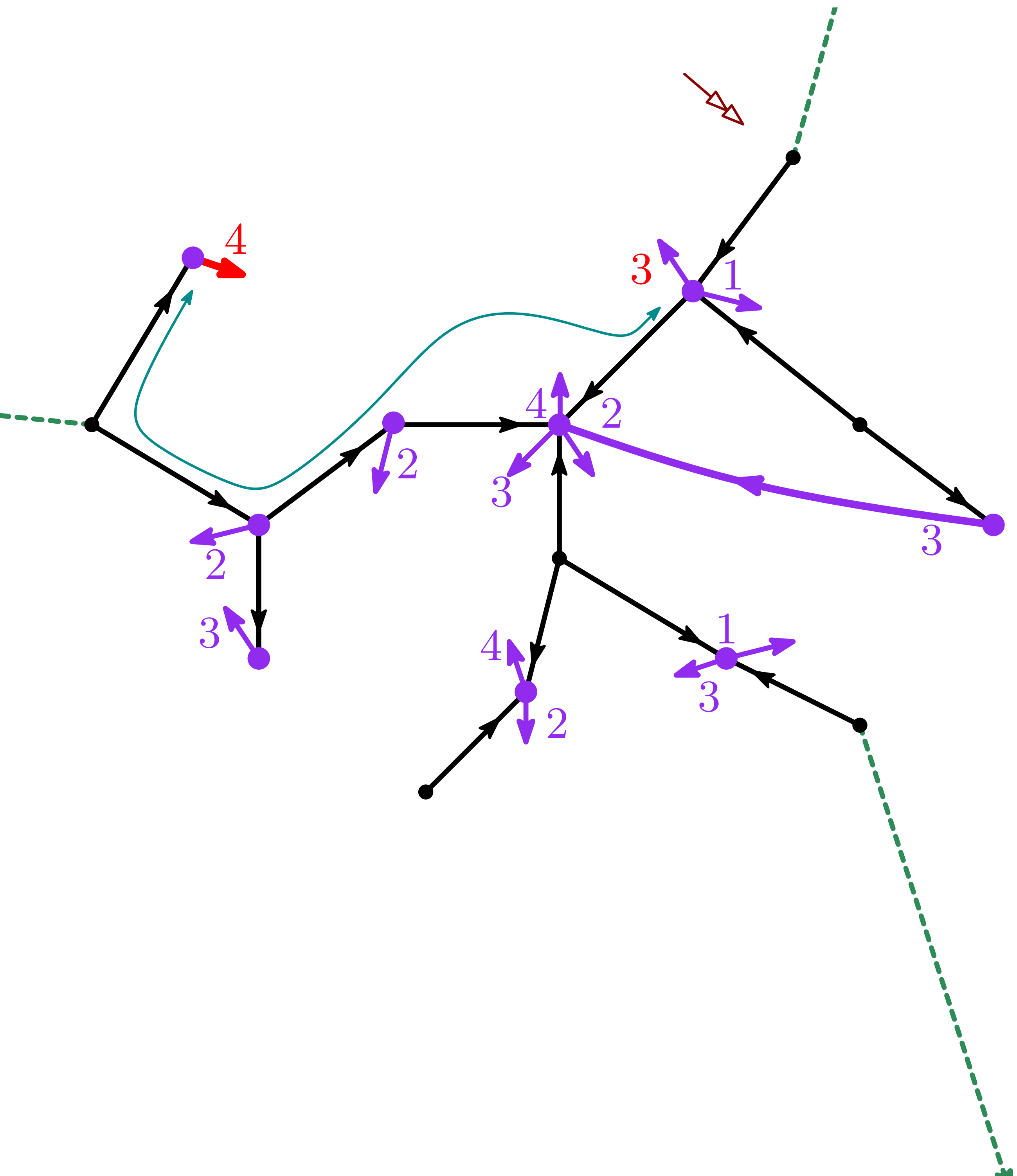

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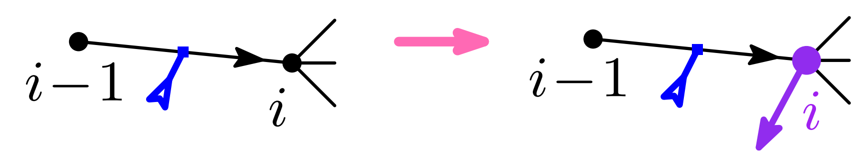

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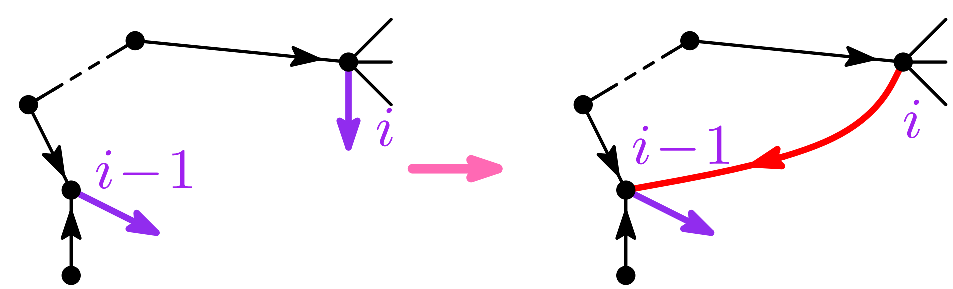

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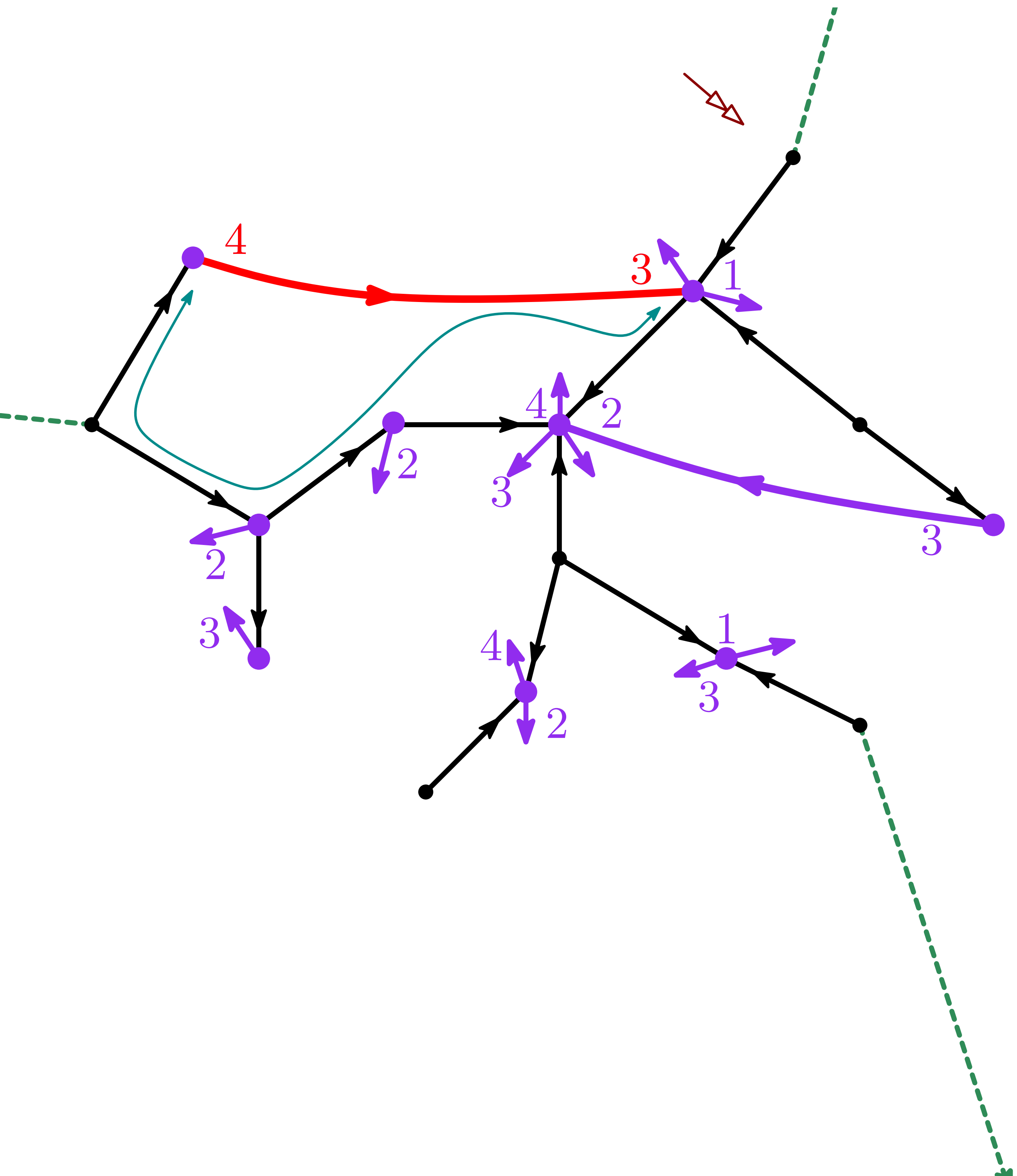

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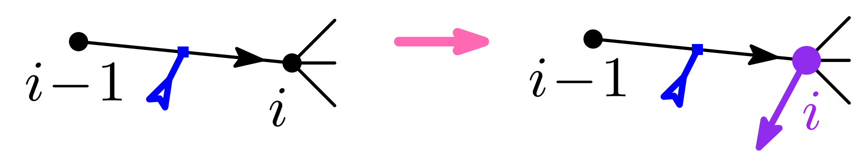

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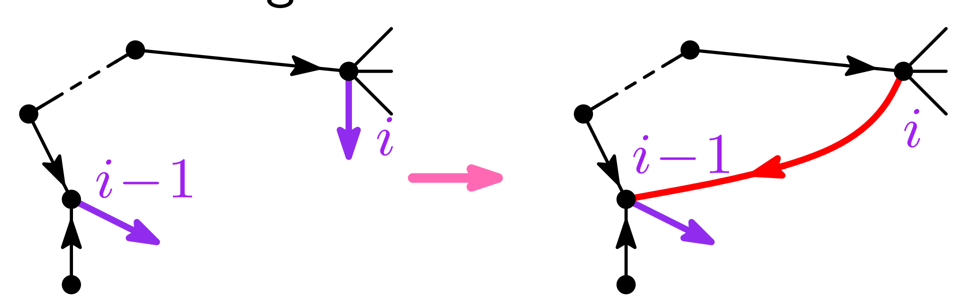

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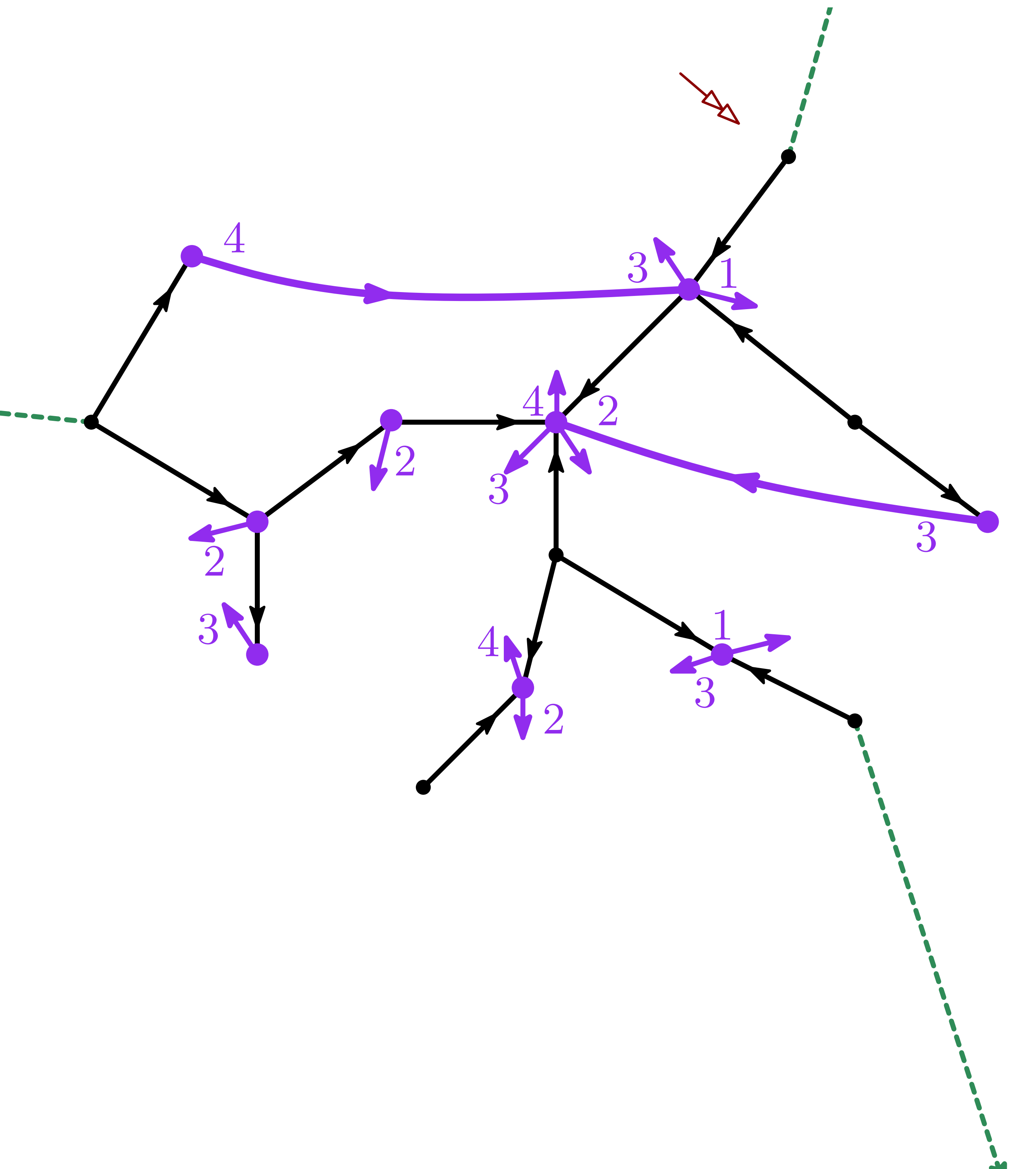

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Labels $\approx$ depth of the face in the cubic map

Unmatched stems = last 0, 1 and 2 corners

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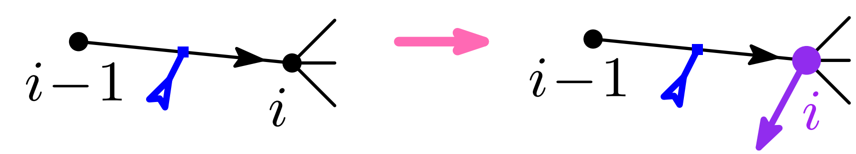

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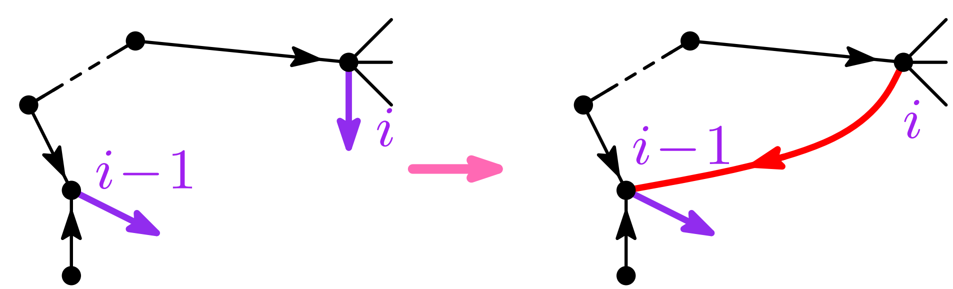

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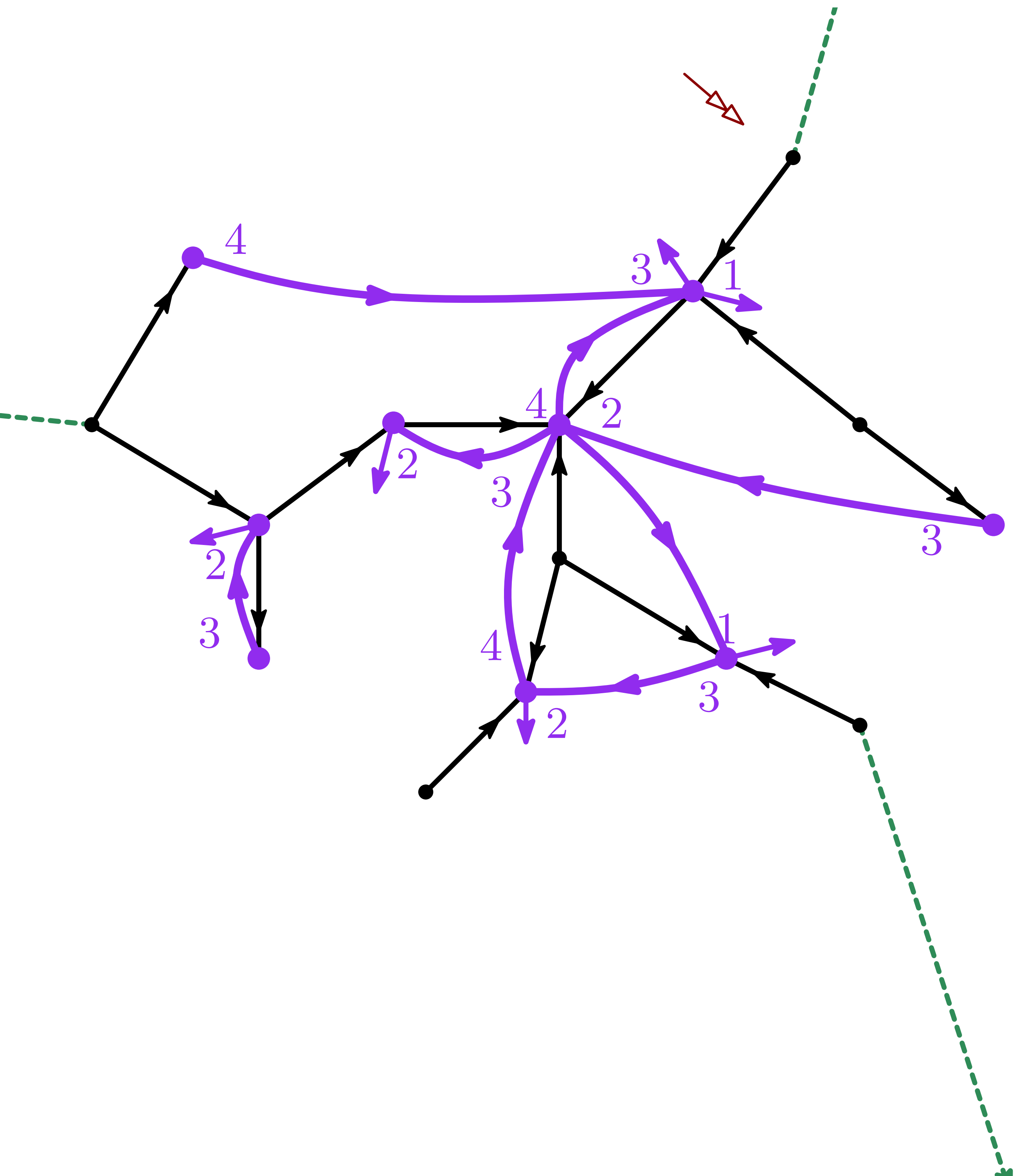

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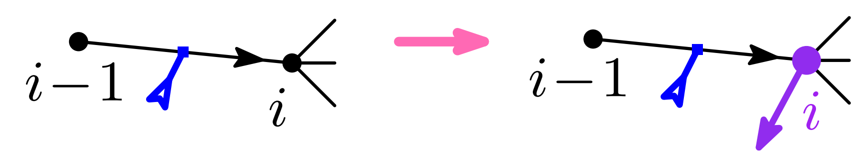

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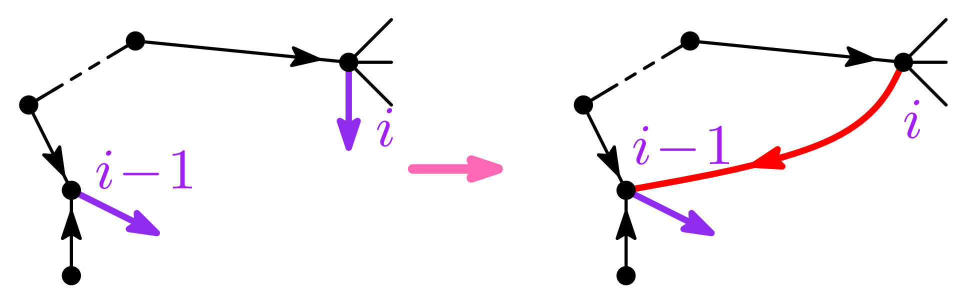

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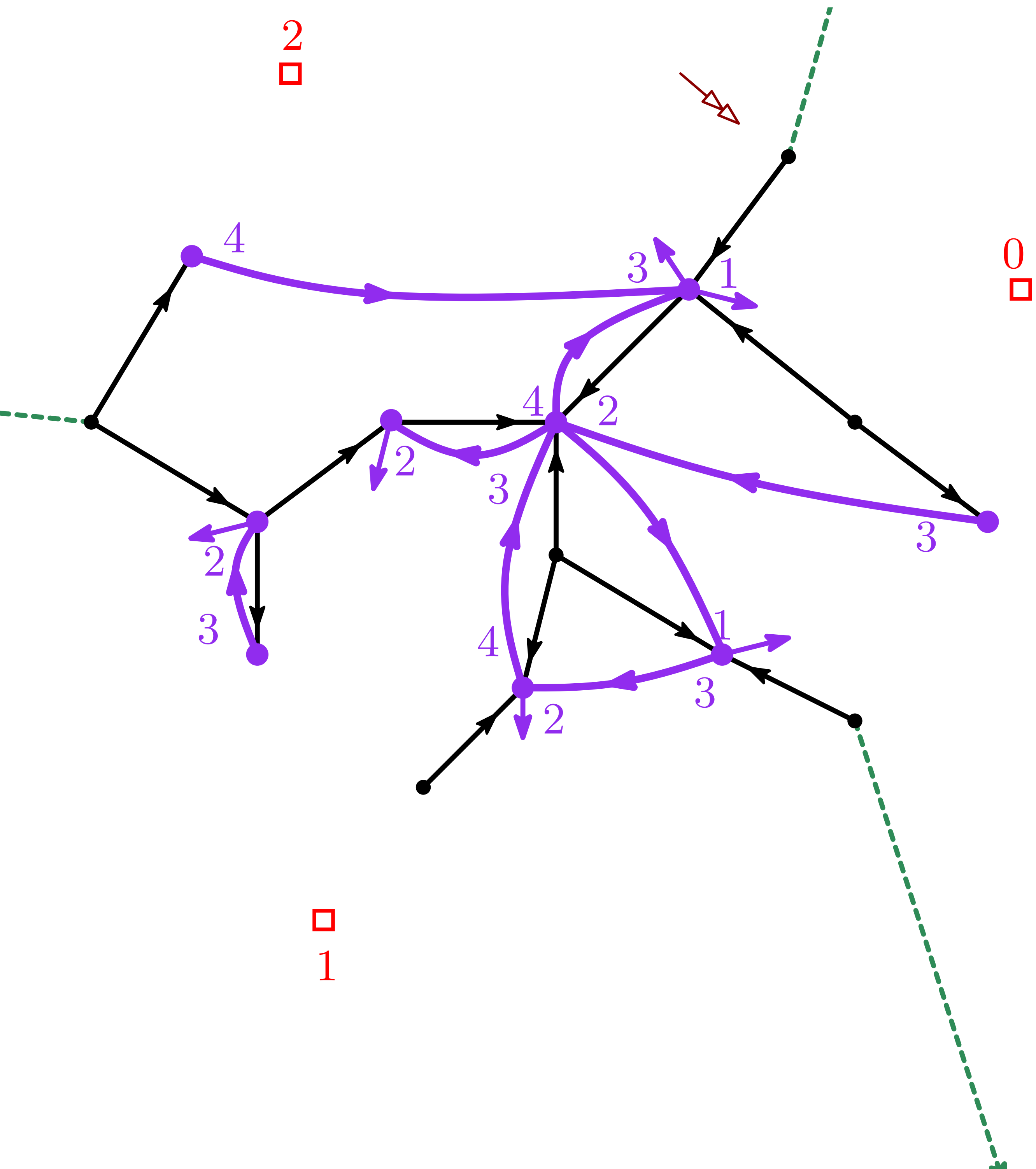

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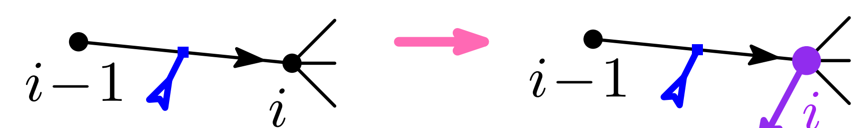

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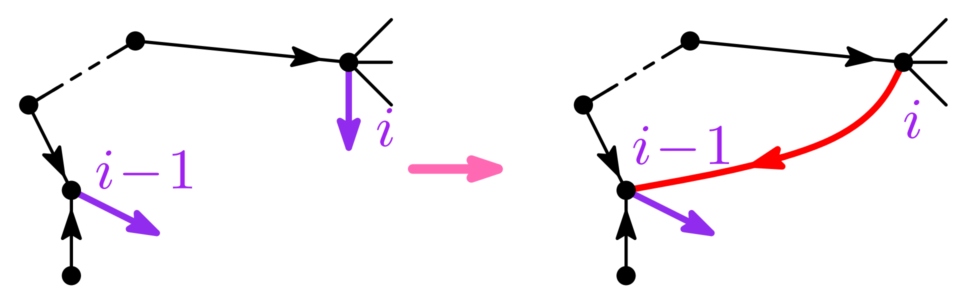
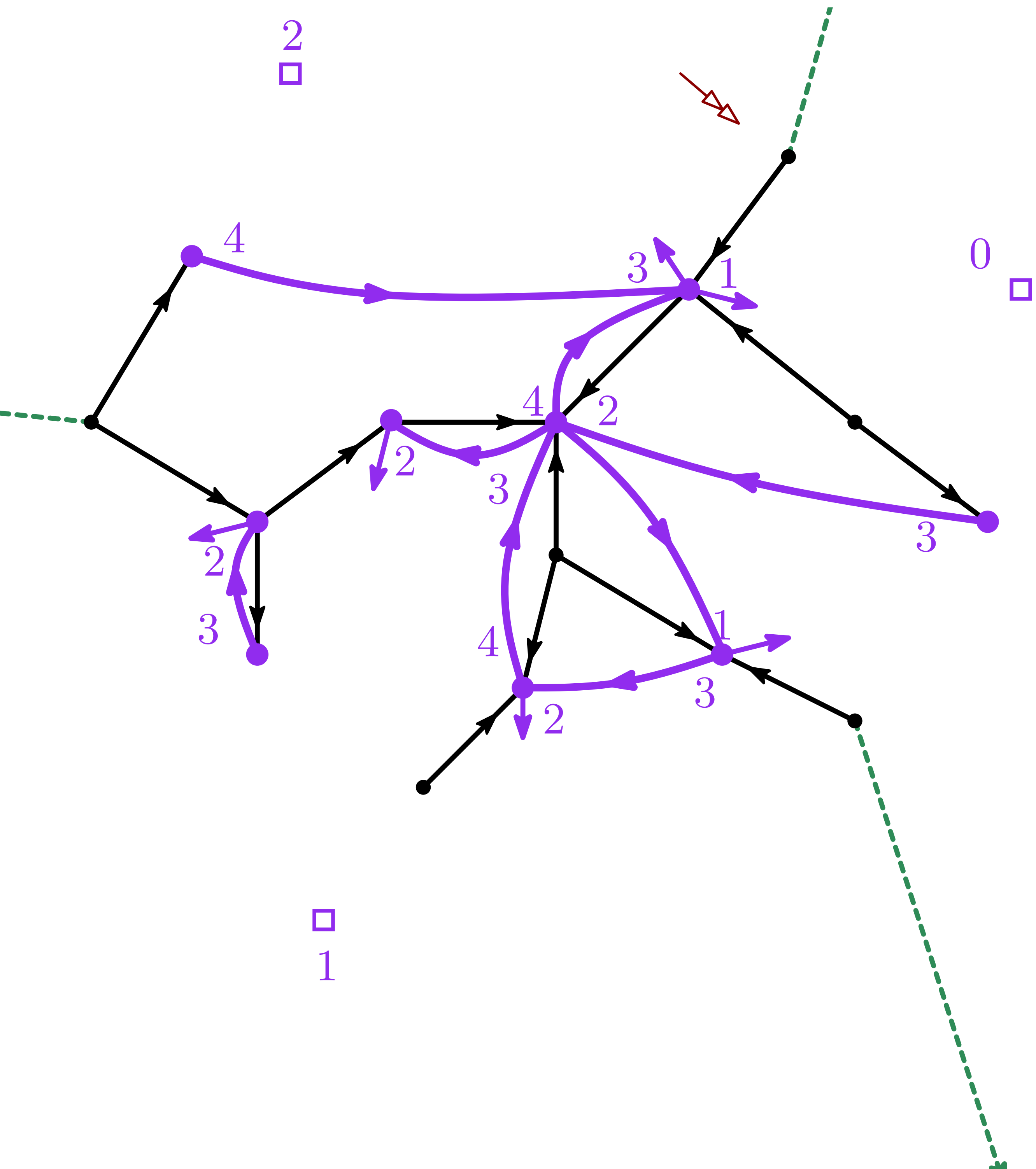


- Add 3 vertices and close the remaining opening stems in the same way.

Same bijection with labels



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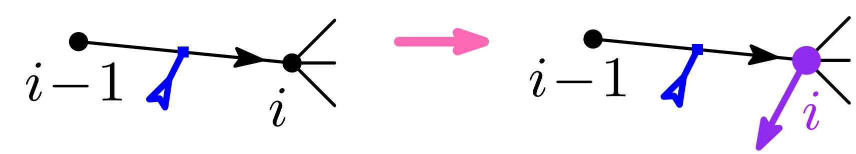
- After a closing stem : decrease by 1.

Labels $\approx$ depth of the face in the cubic map

Unmatched stems = last 0, 1 and 2 corners

- Apply the following local rule :



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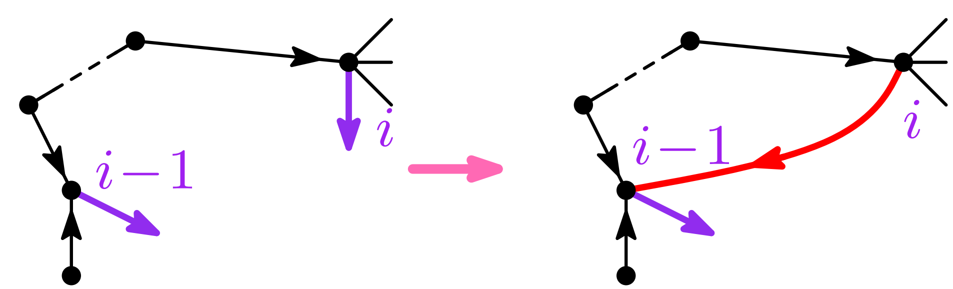

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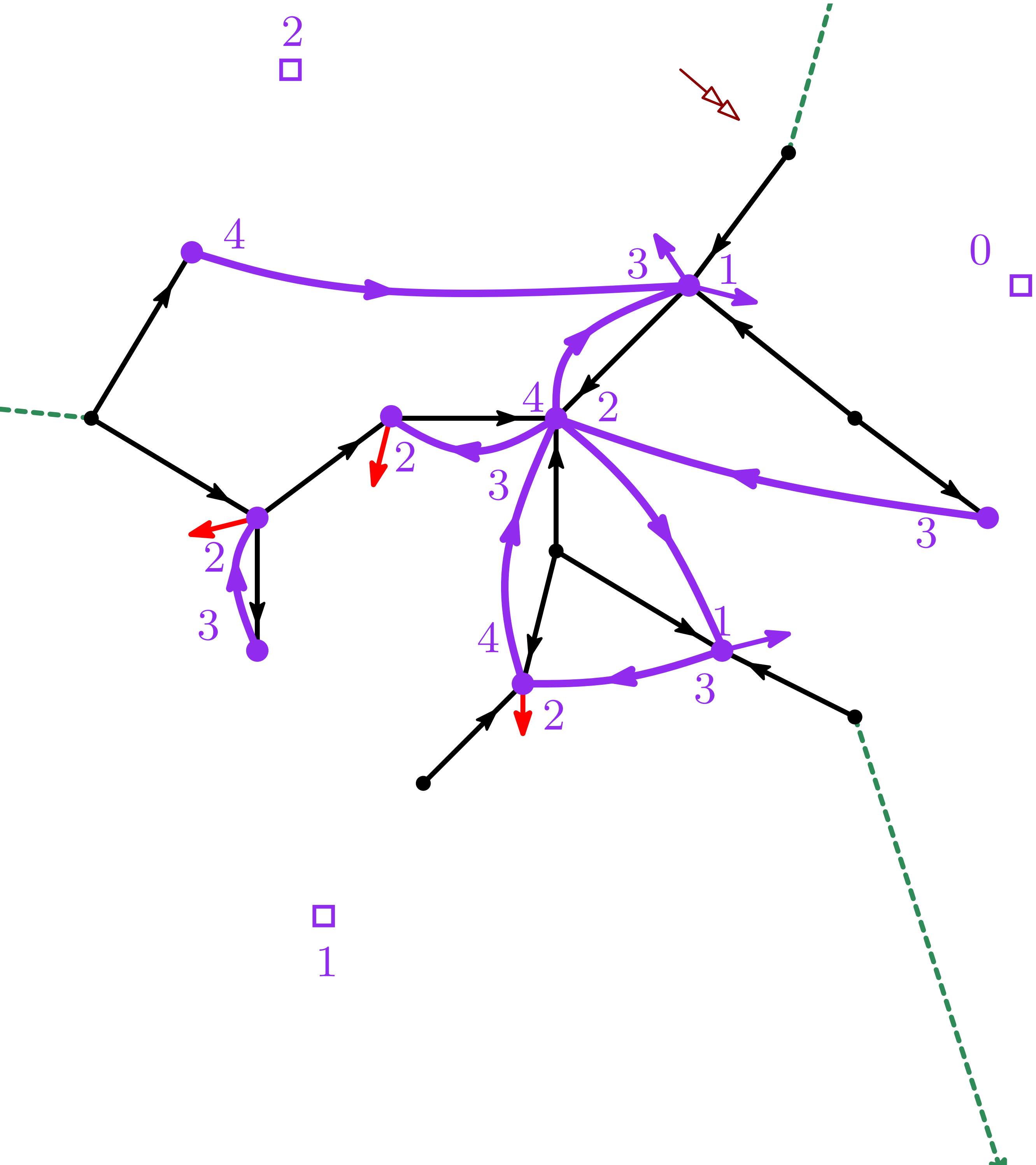

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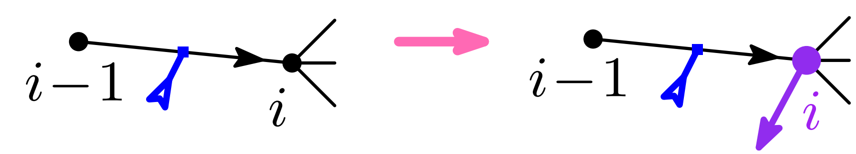

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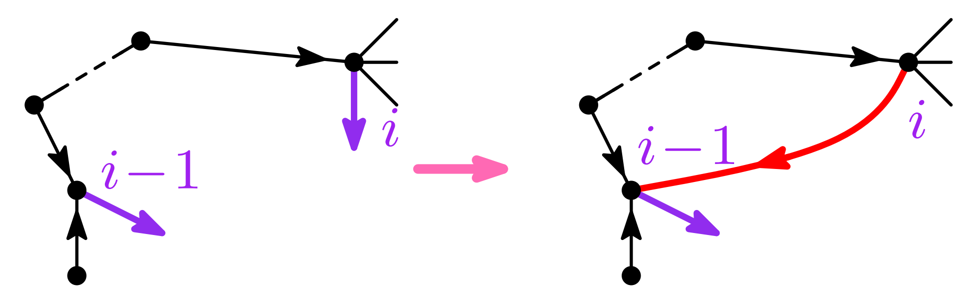

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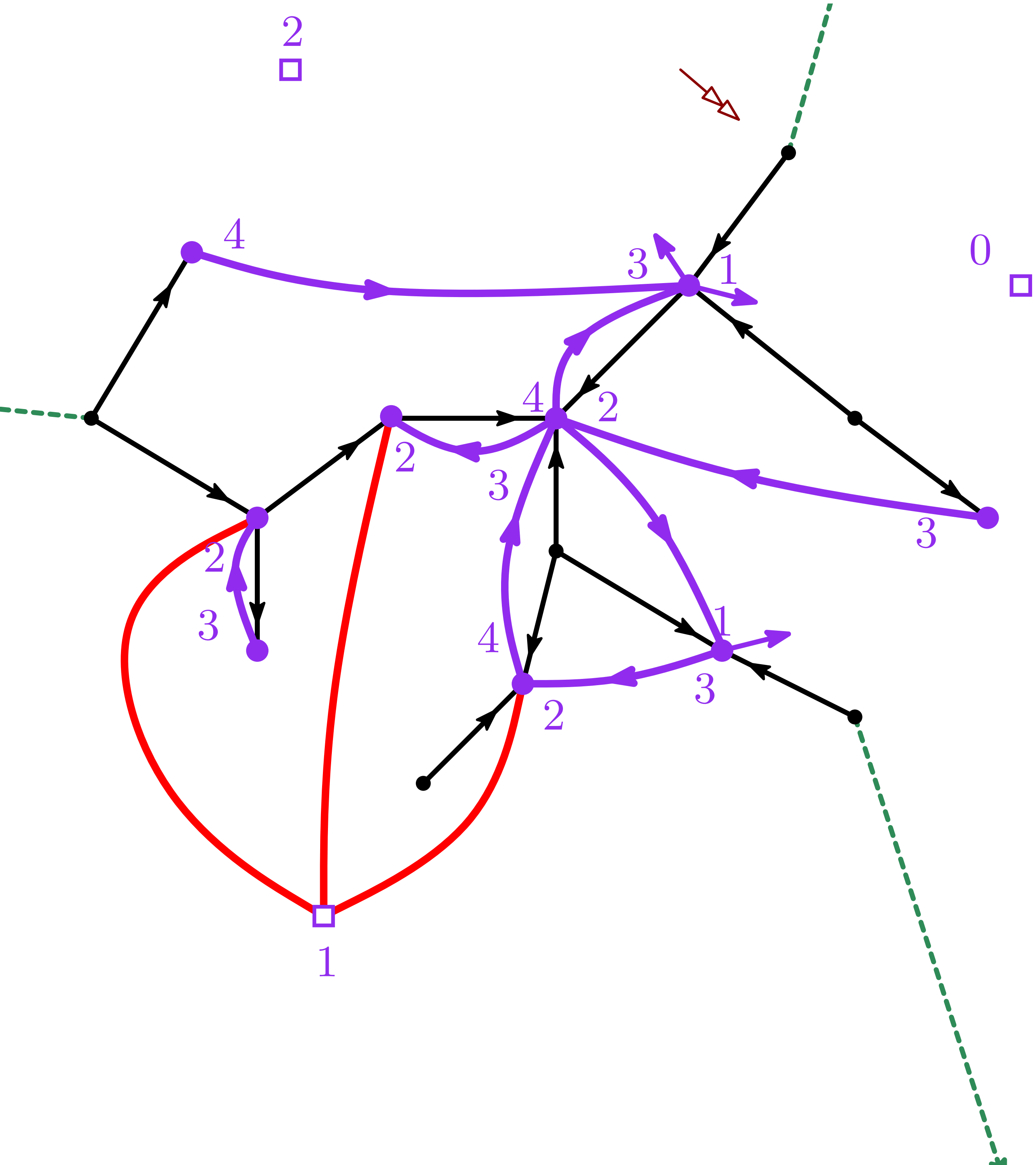

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Labels ≈ depth of the face in the cubic map

Unmatched stems = last 0, 1 and 2 corners

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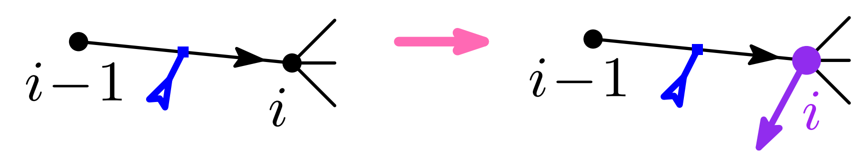

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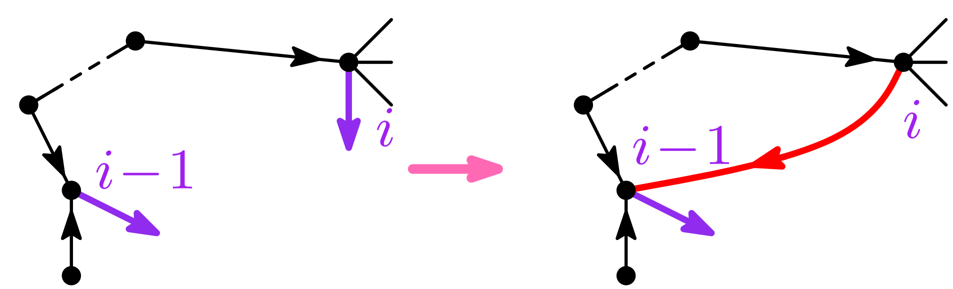

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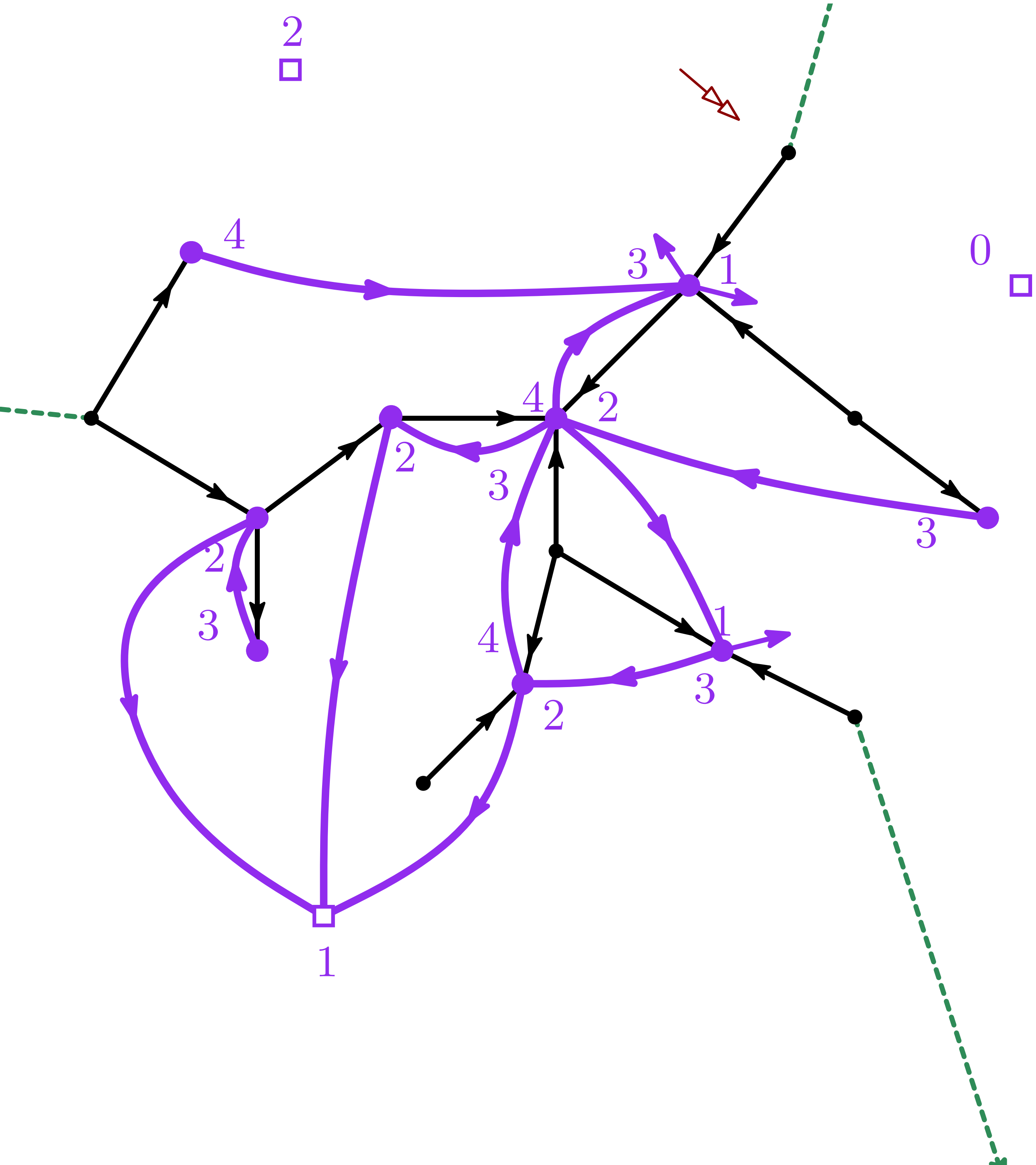

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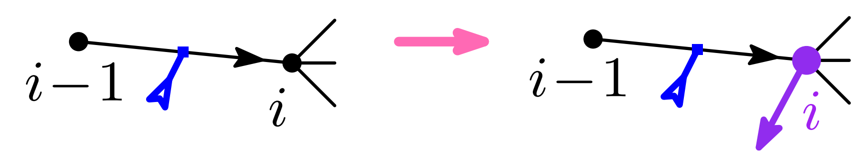
Labels $\approx$ depth of the face in the cubic map

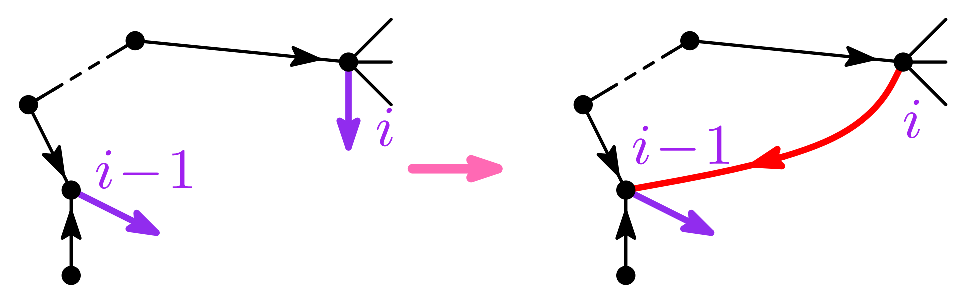
Unmatched stems = last 0, 1 and 2 corners

- Apply the following local rule :



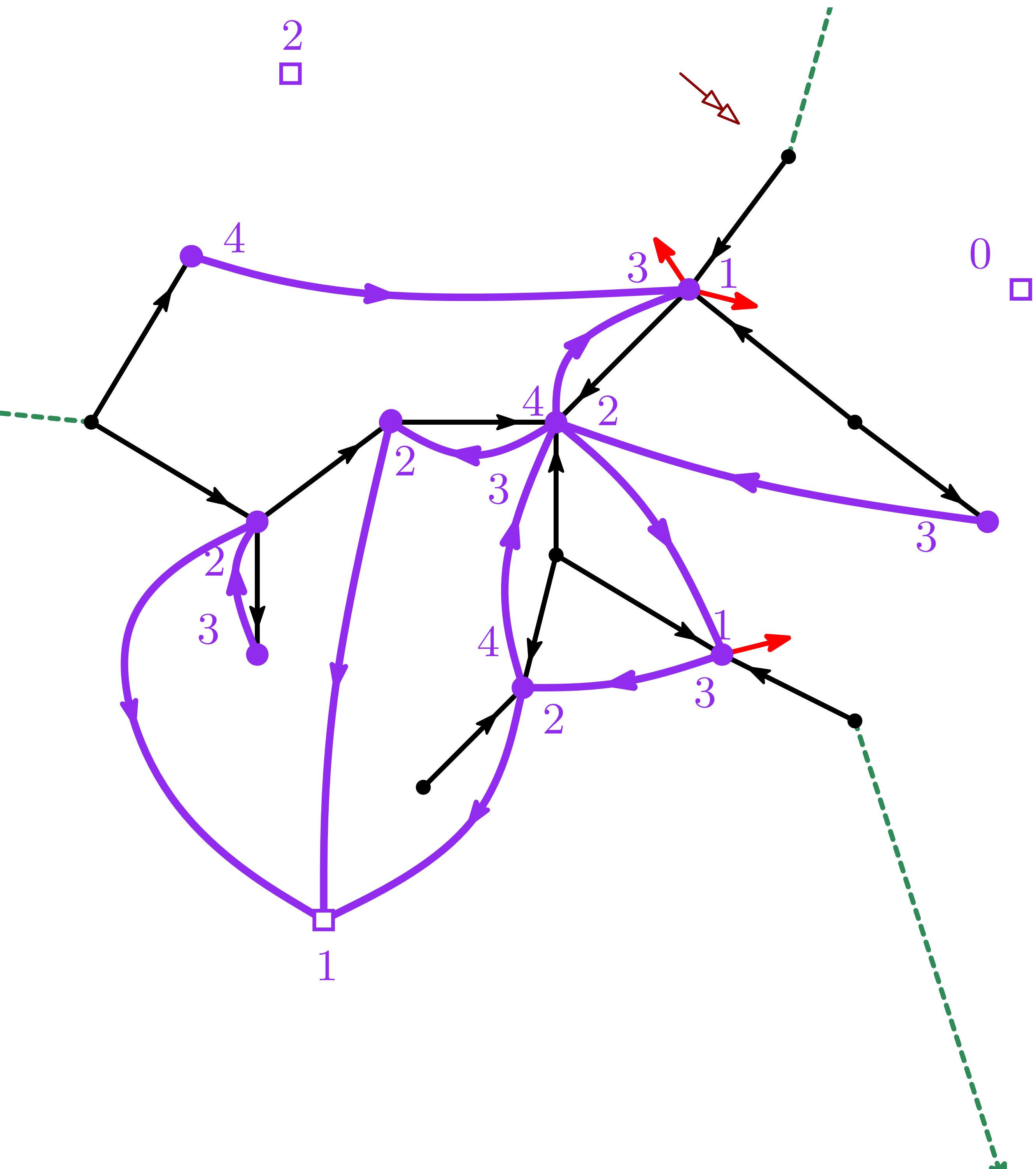
(= add a → before each descent and color the corresponding corner and vertex)

- Erase all non-purple and do the following closures



- Add 3 vertices and close the remaining opening stems in the same way.

Same bijection with labels



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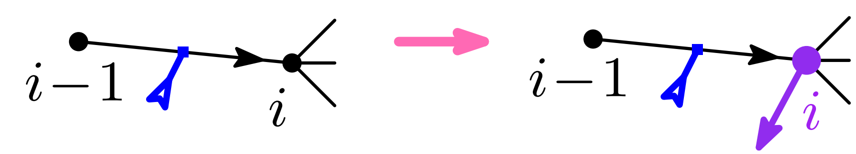

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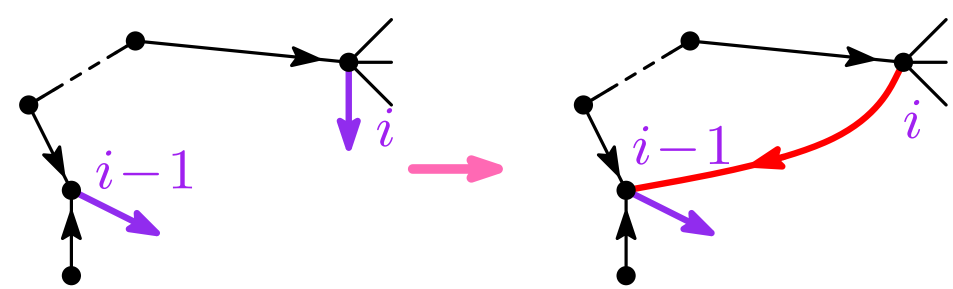

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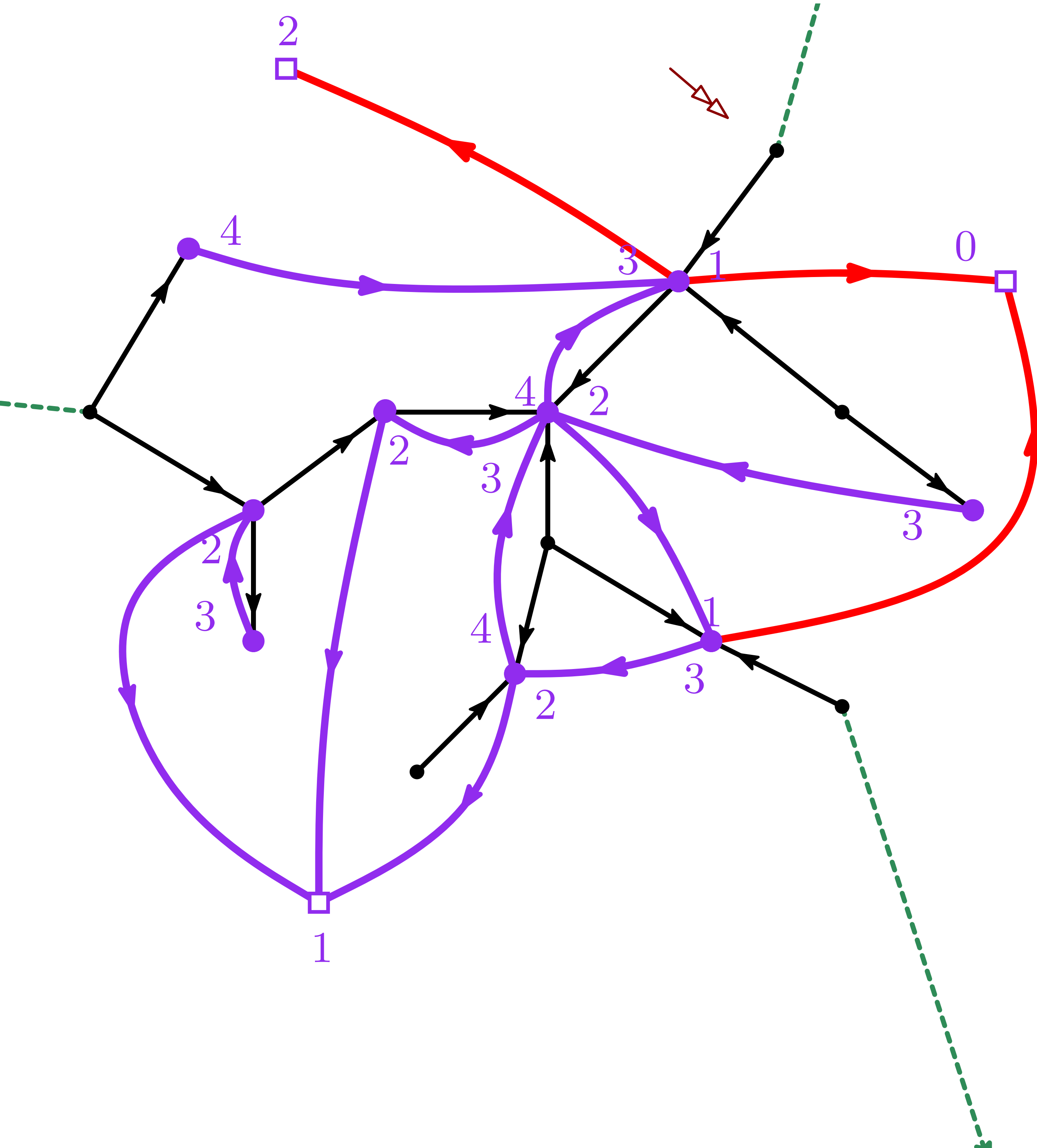

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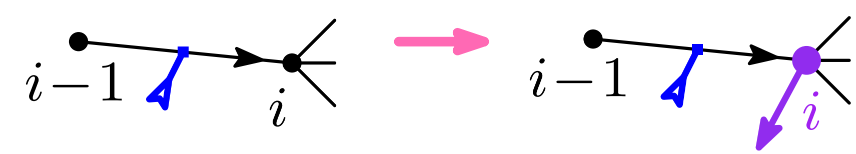

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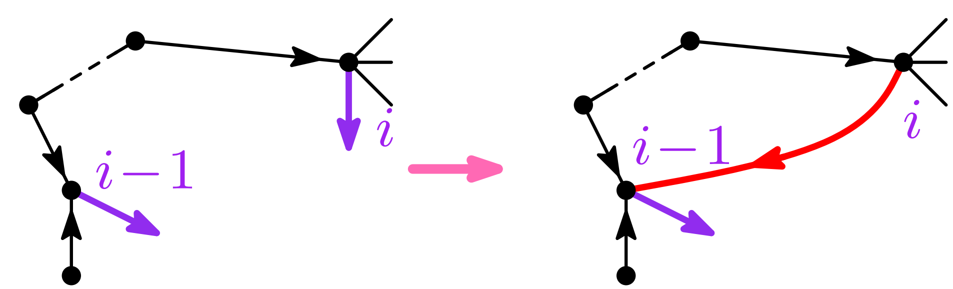

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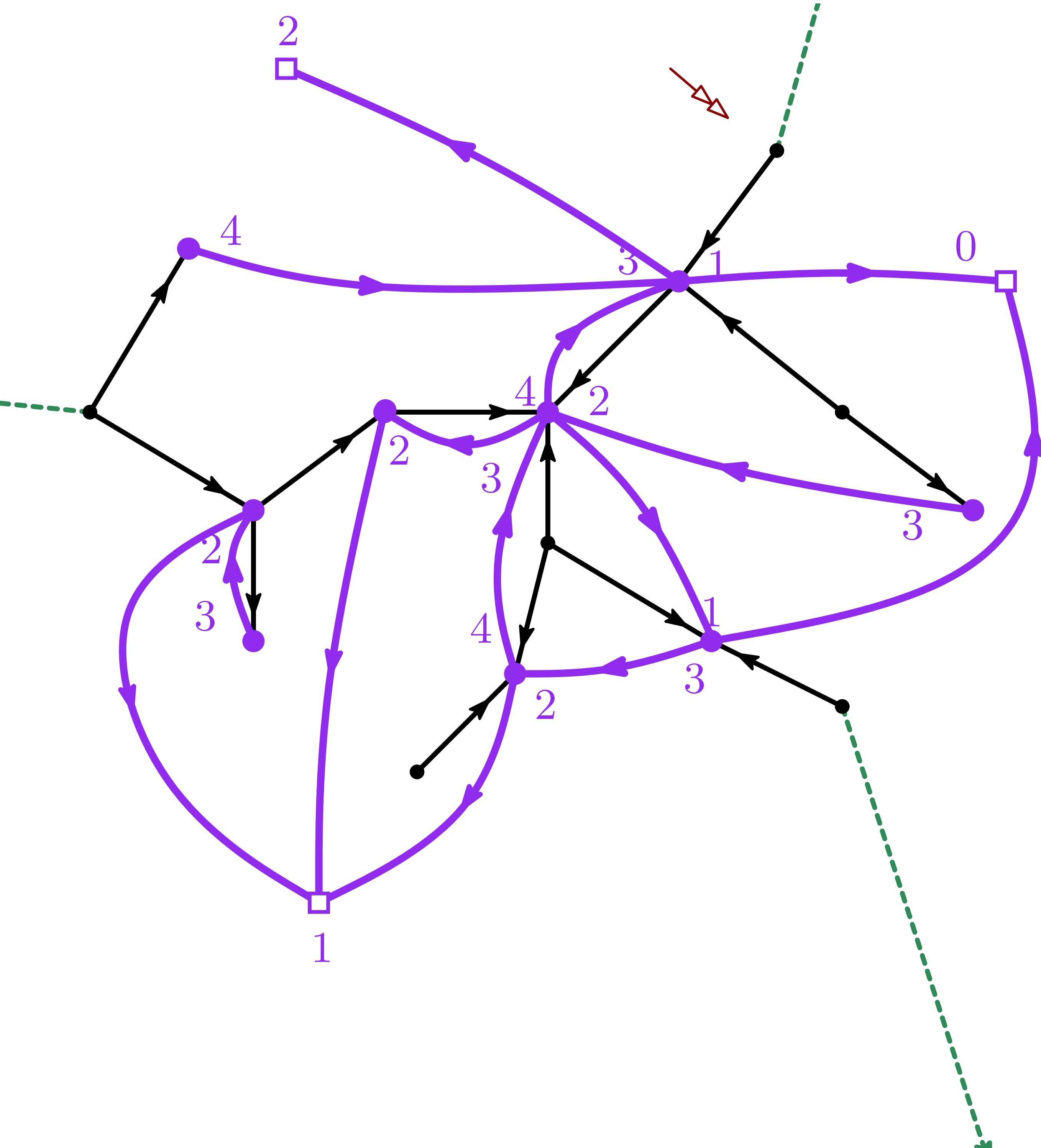

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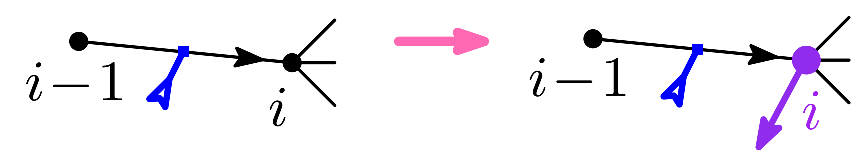

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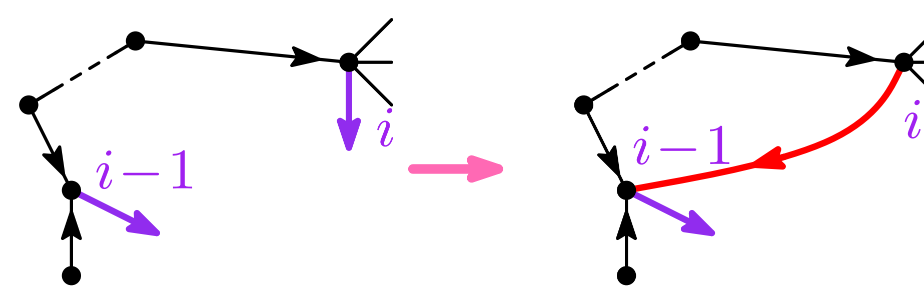

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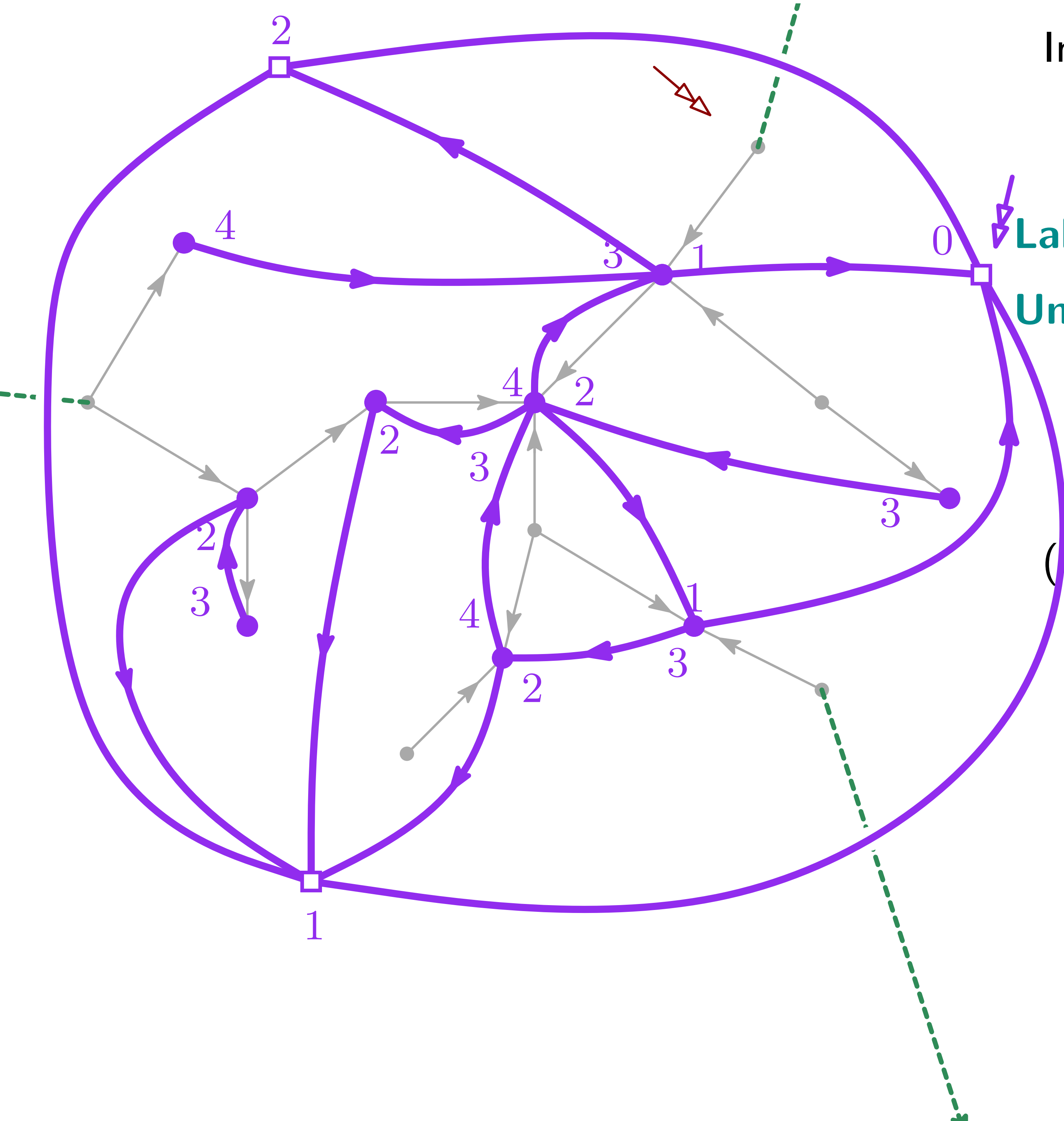

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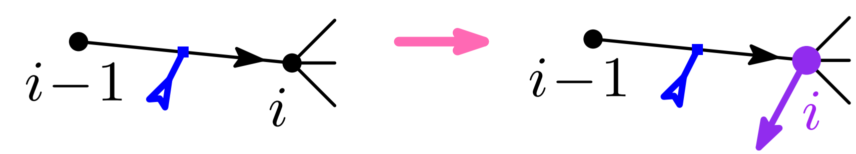

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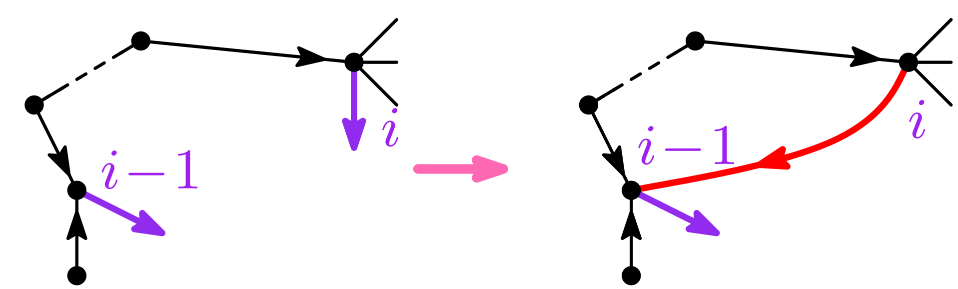

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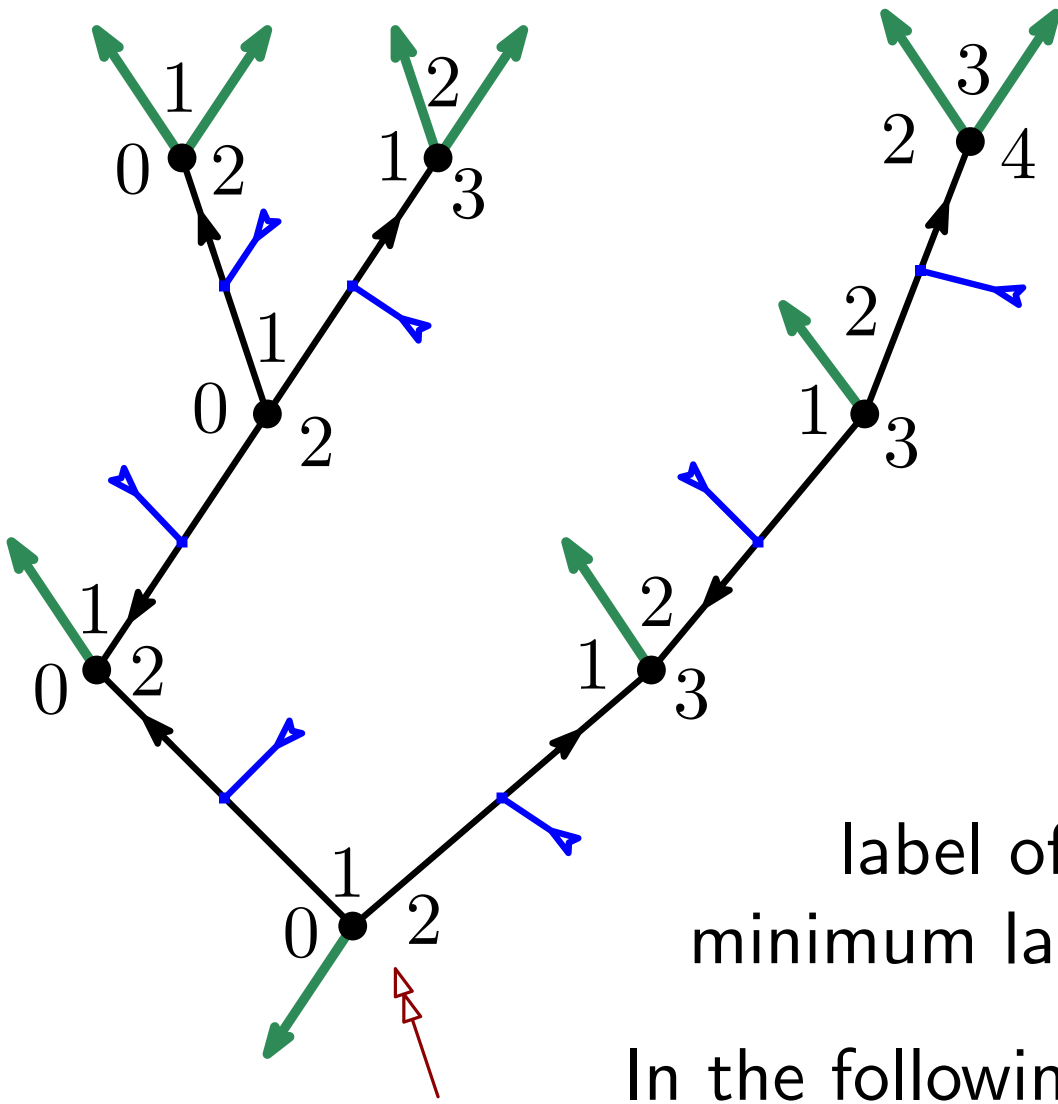
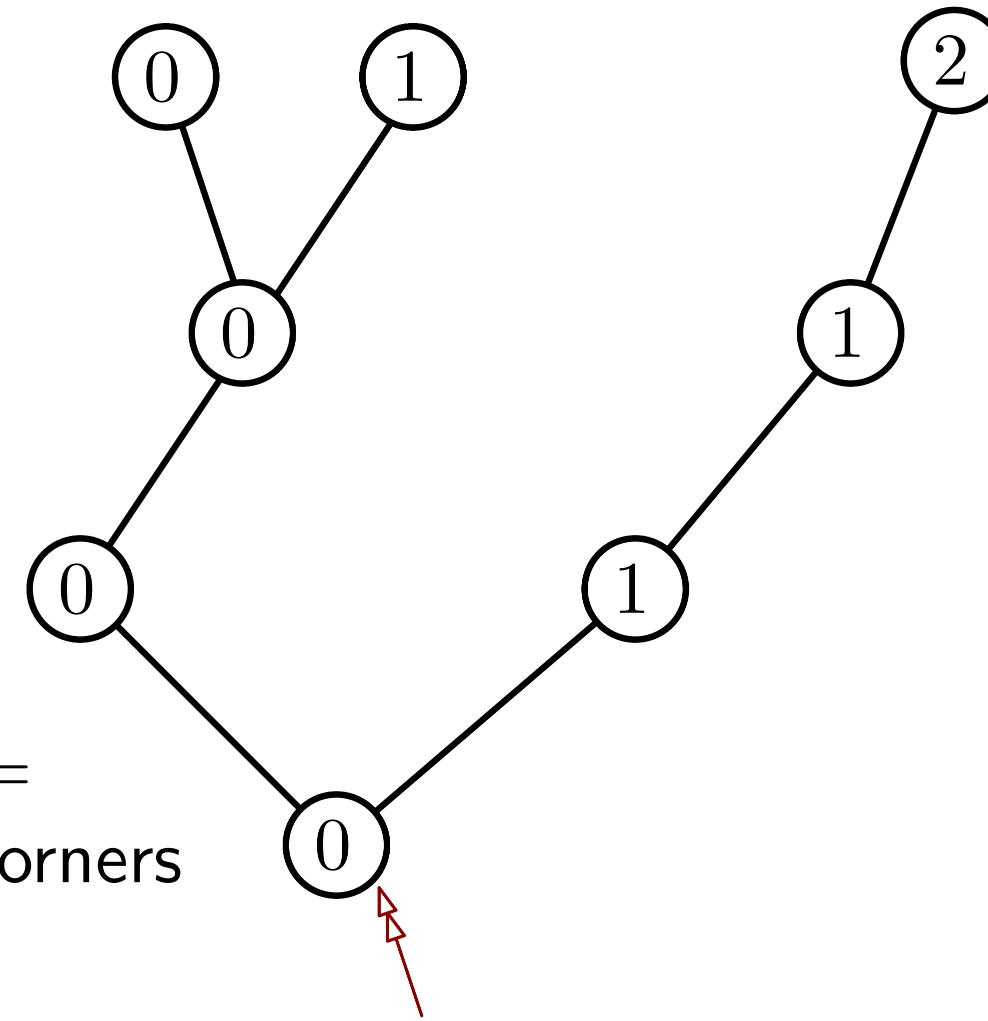
From blossoming trees to labeled trees

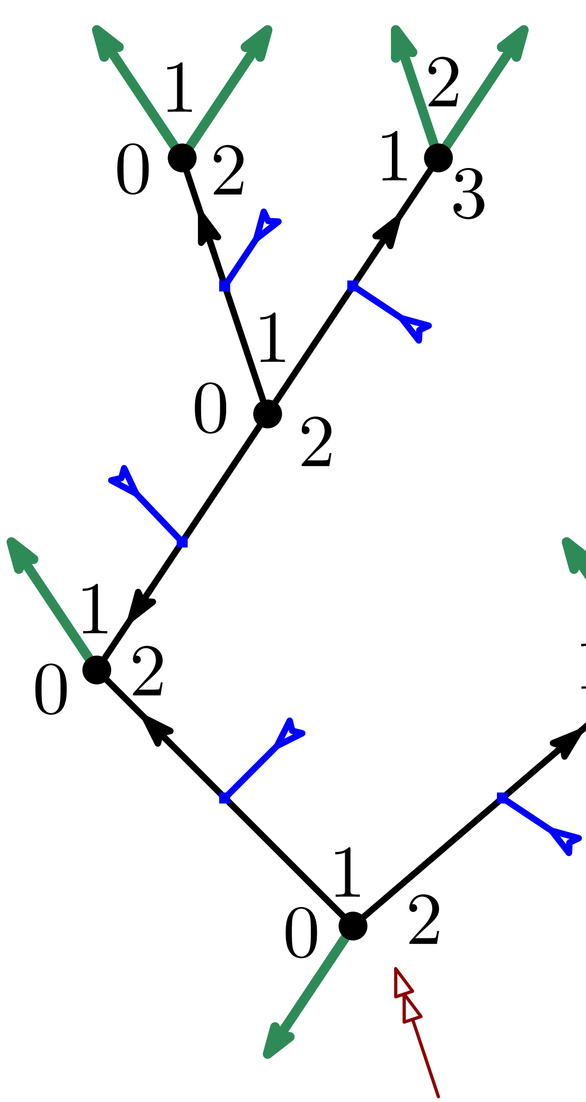




label of a vertex =
minimum label of its corners

In the following:
Labels gives approximate
distances to the root **in the map**

From blossoming trees to labeled trees



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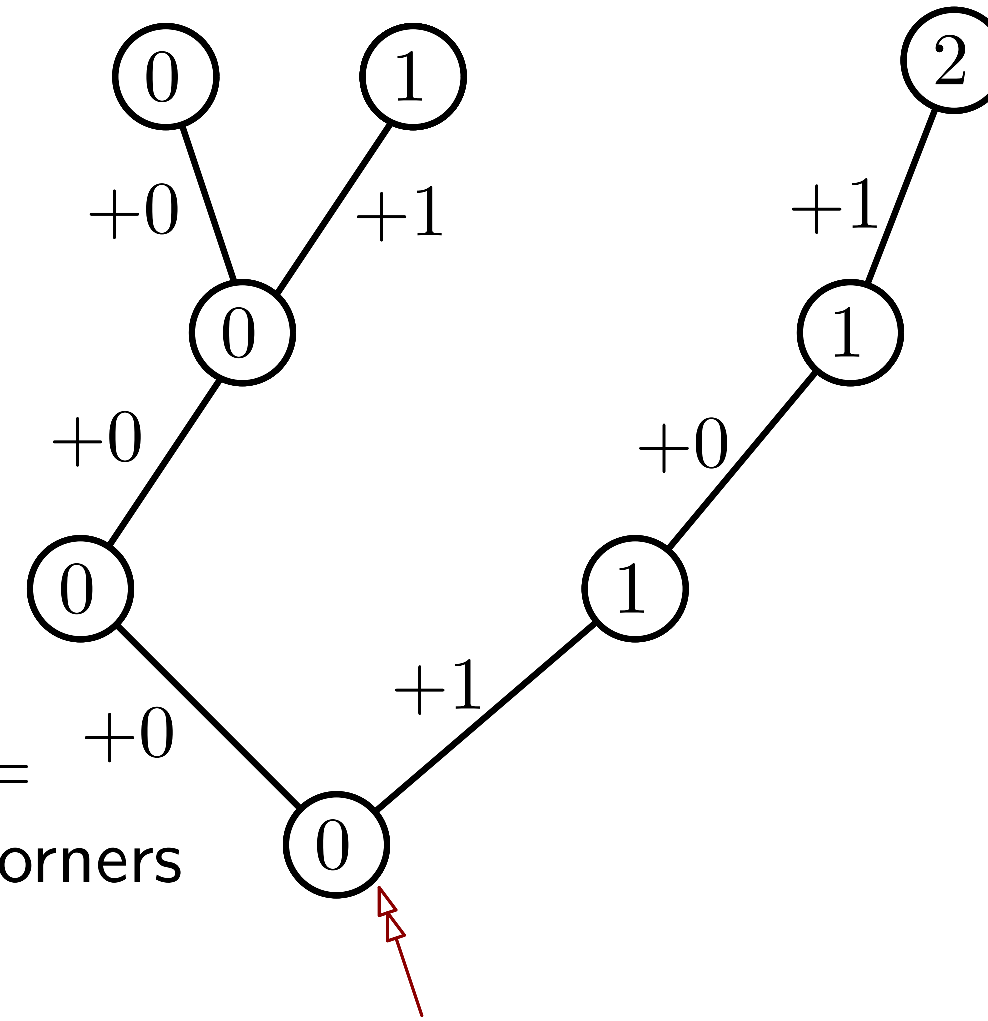

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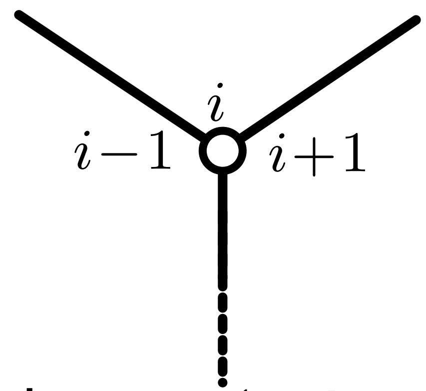
Around each vertex :



For instance, for a node of degree 1, 4 possibilities:

From blossoming trees to labeled trees

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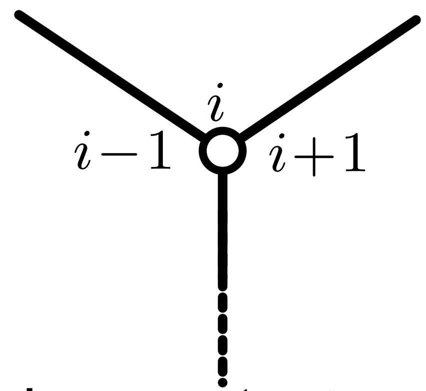

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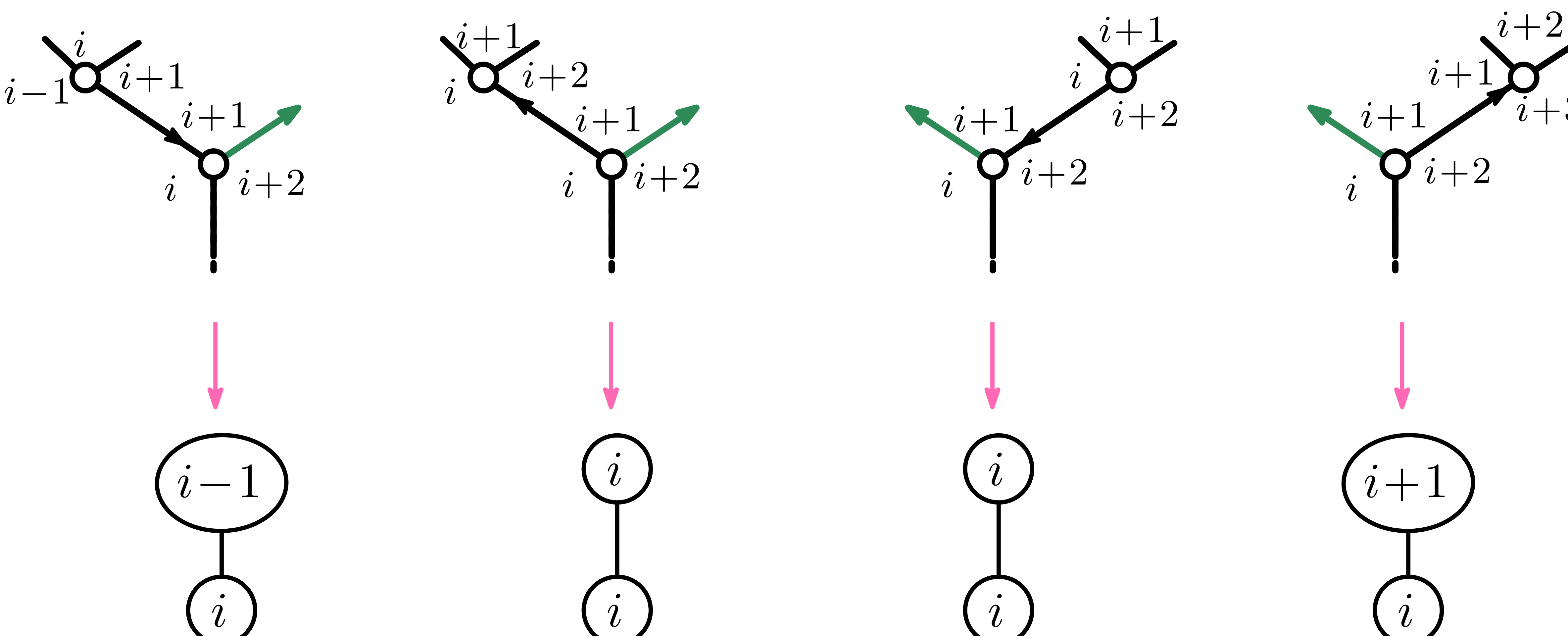
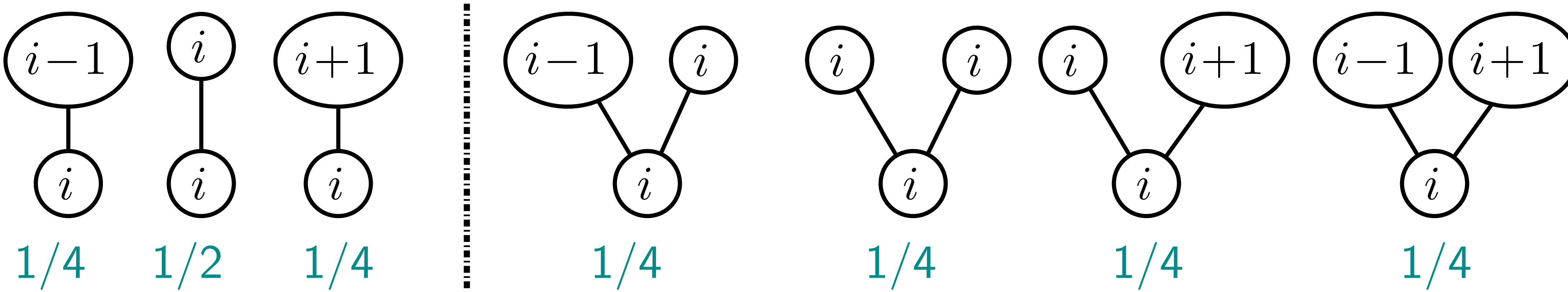


From blossoming trees to labeled trees

To do that :
- encode the maps by some trees.
- **study the limits of trees**,
- interpret the distance in the maps by some function of the tree.

- Labeled tree = GW binary trees + random displacements on edges



exactly the setting of [Marckert '08]:
convergence to the Brownian snake with the labels normalized by $(2n)^{1/4}$

Convergence of labeled trees

Theorem : [Marckert '08]
For a sequence of simple random outer-triangular maps (M_n), the contour and label process of the associated labeled tree satisfie:

$$\left((8n)^{-1/2}C_{\lfloor nt \rfloor}, (1/2n)^{1/4}\tilde{Z}_{\lfloor nt \rfloor}\right)_{0\le t\le 1} \xrightarrow[n\to\infty]{(d)} (e_t, Z_t)_{0\le t\le 1},$$

Contour and label processes of a labeled tree

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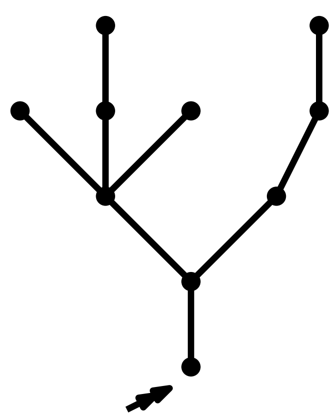

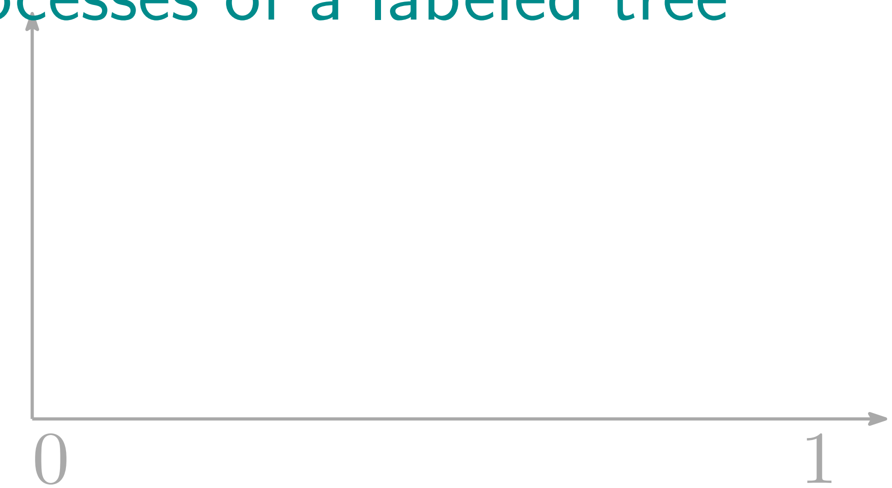

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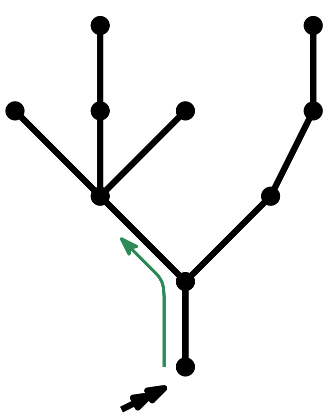

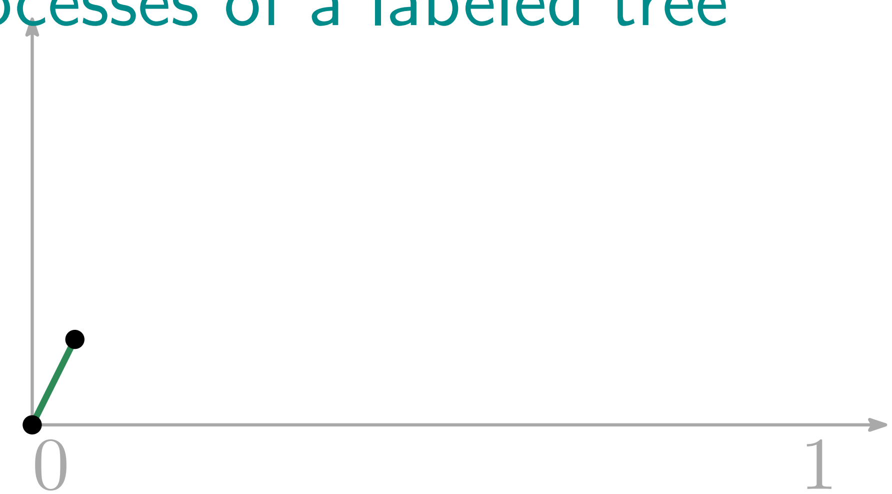

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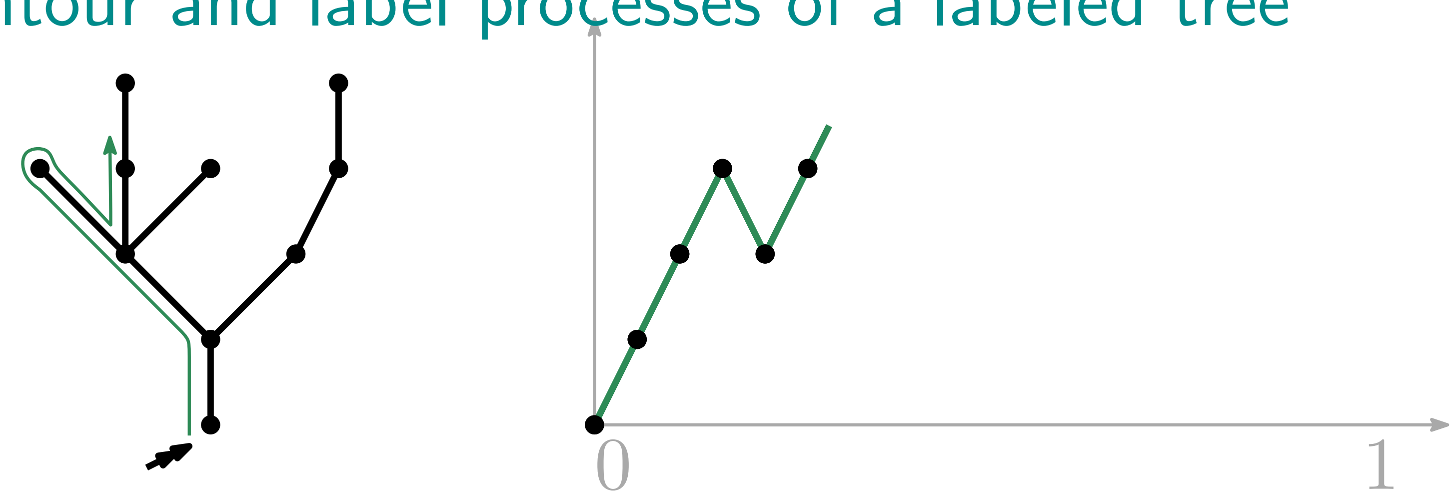

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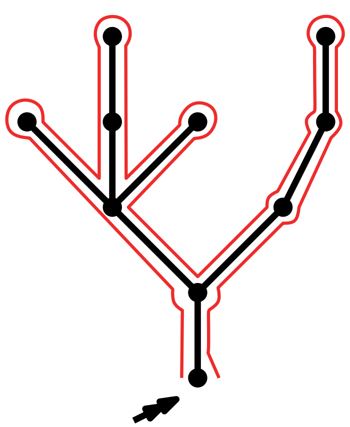

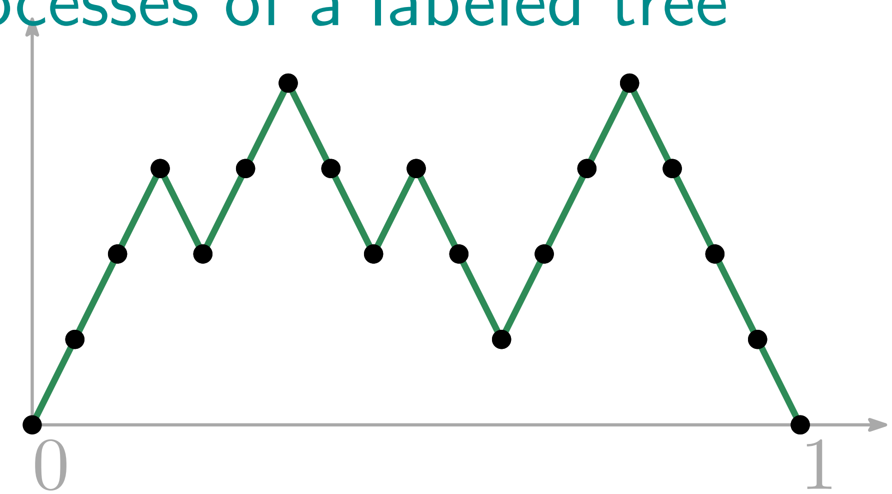

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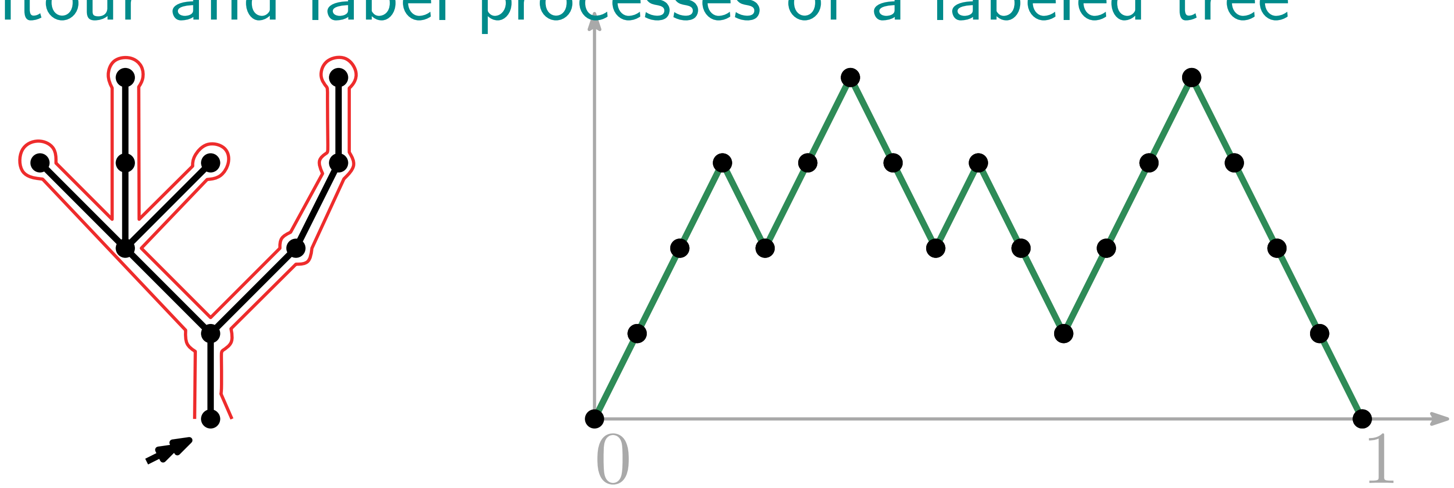

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Contour and label processes of a labeled tree

i and j = same vertex of T iff $C_n(i) = C_n(j) = \min_{i\leq k\leq j} C_n(k)$

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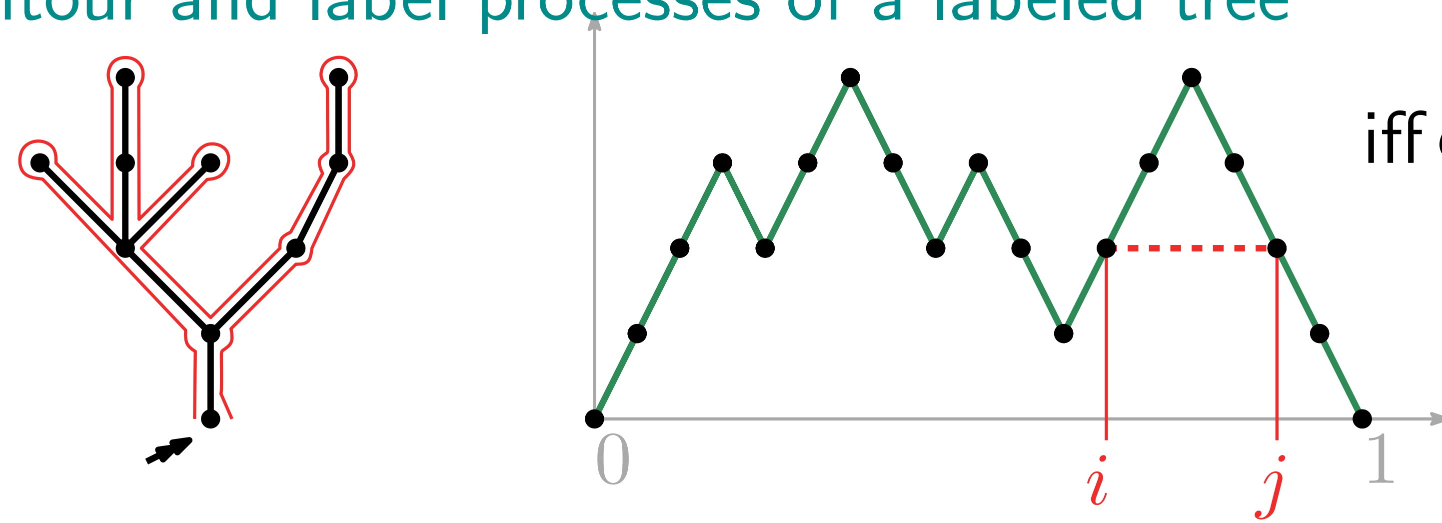

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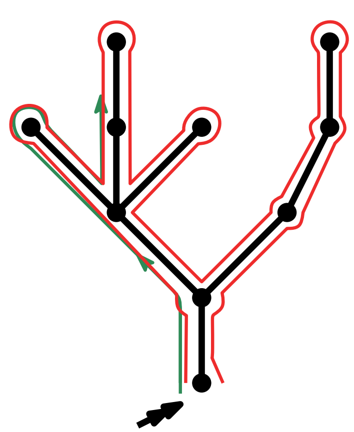

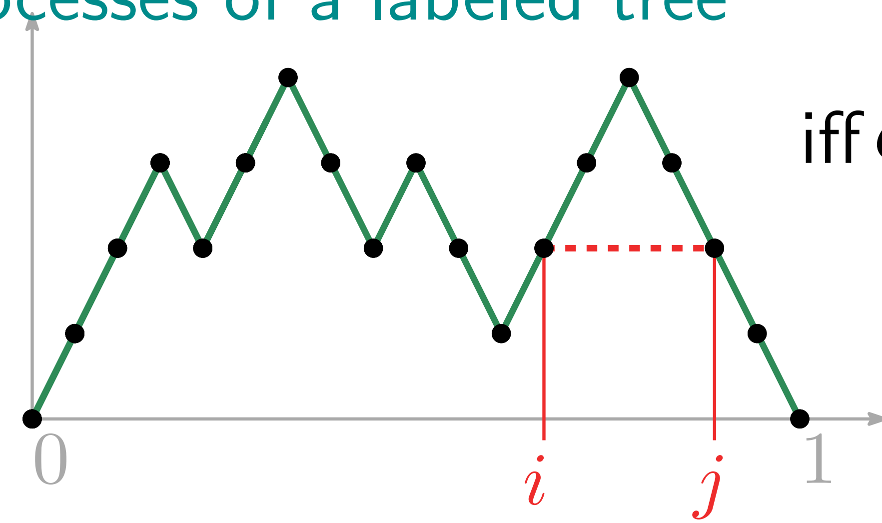

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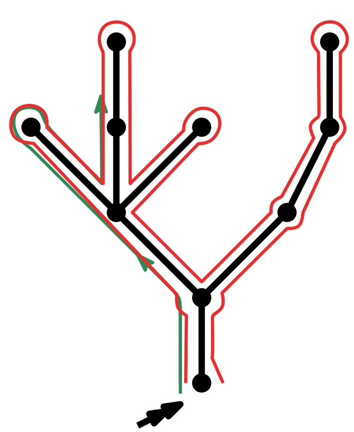

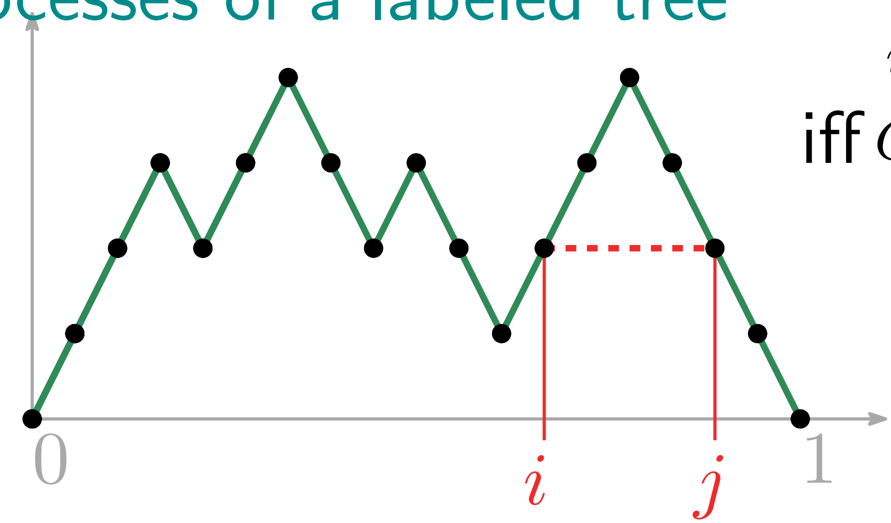

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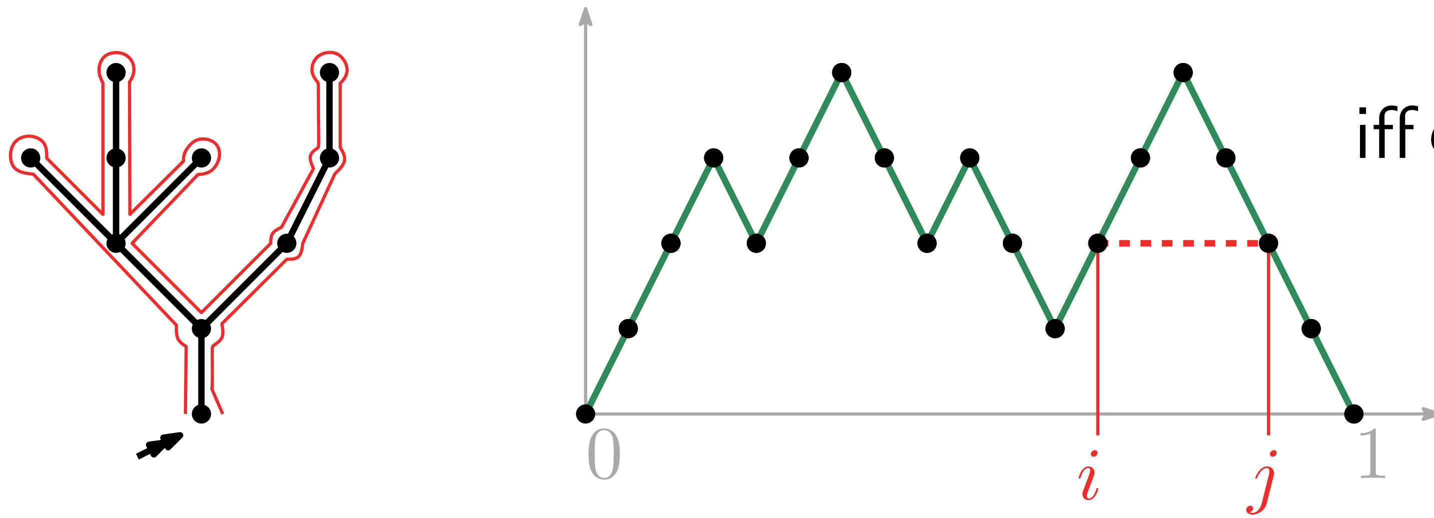
Brownian snake $(e_t, Z_t)_{0 \le t \le 1}$

1st step : the Brownian tree [Aldous]



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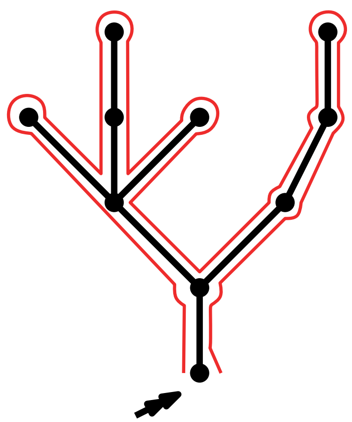

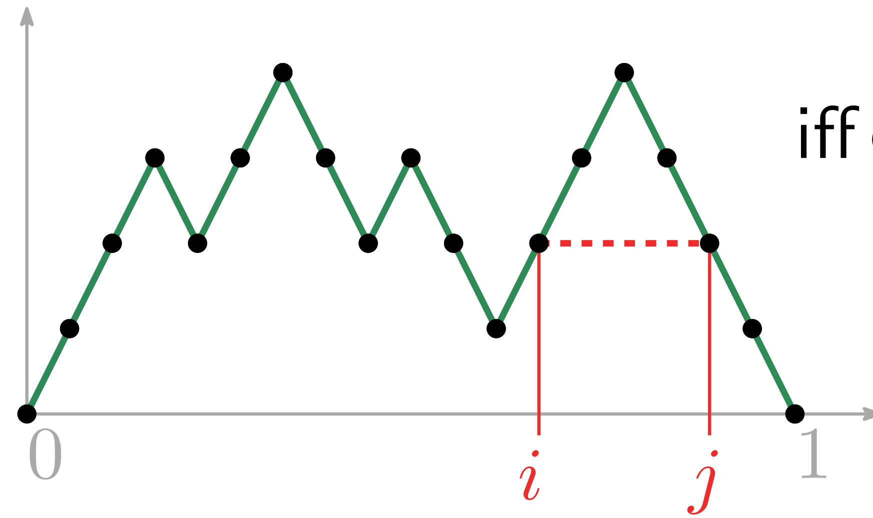

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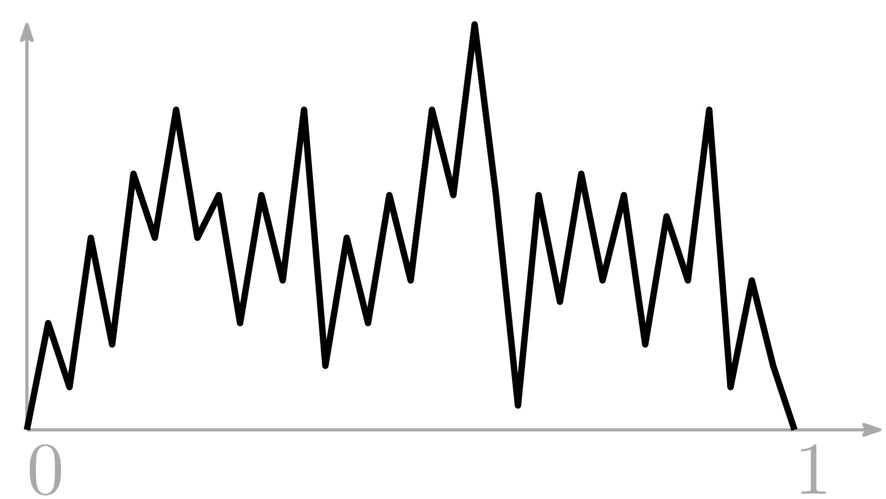

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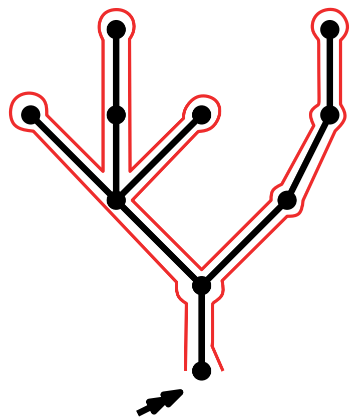

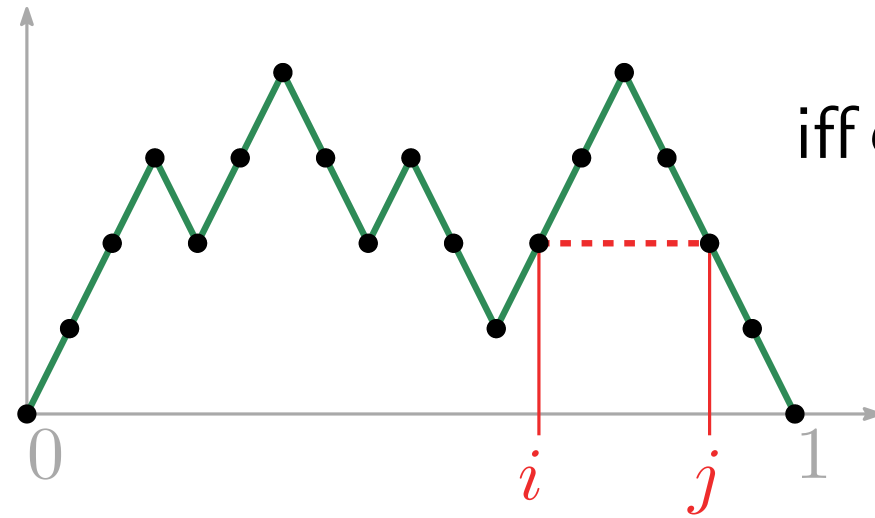

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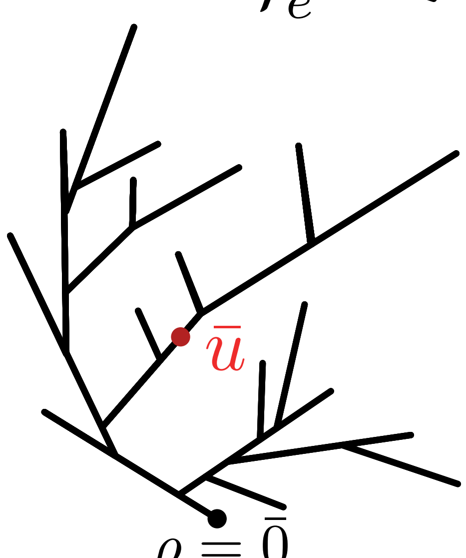

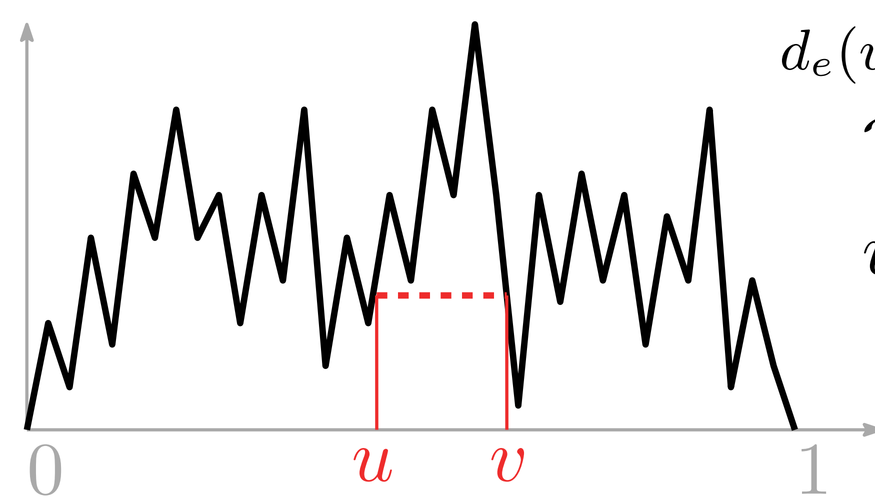

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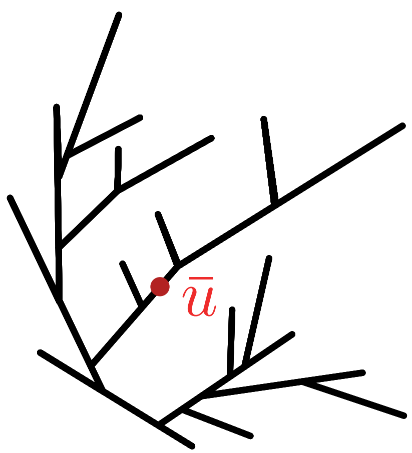

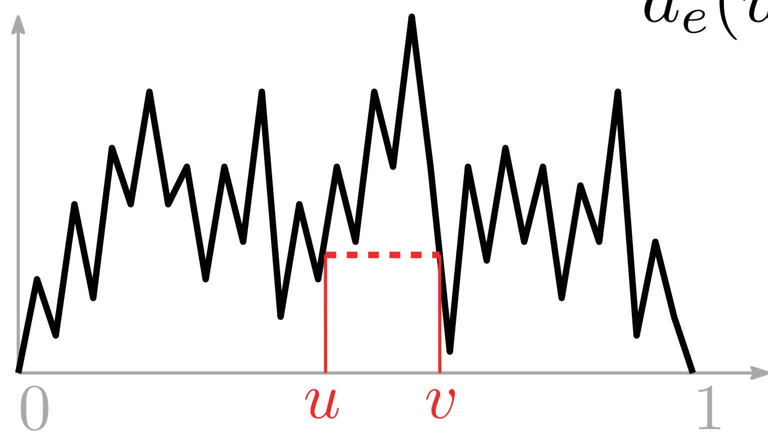

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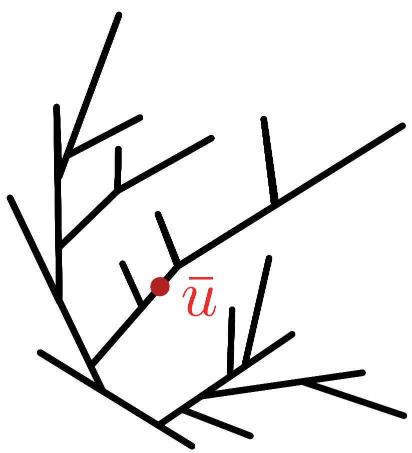

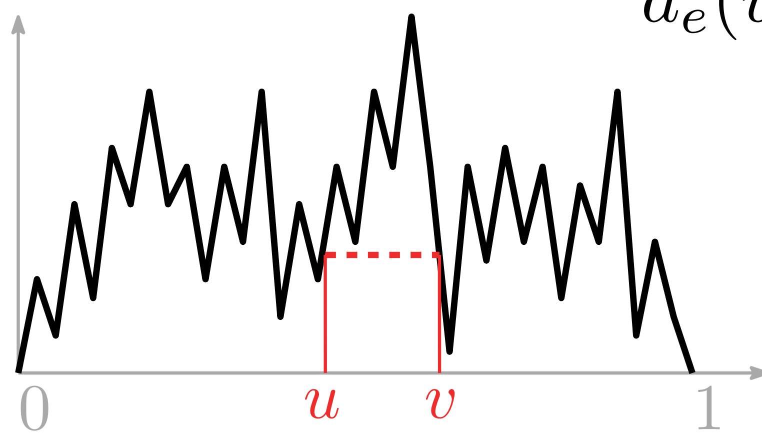

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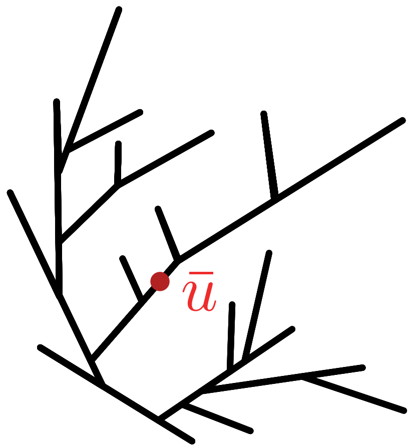
2nd step : Brownian labels

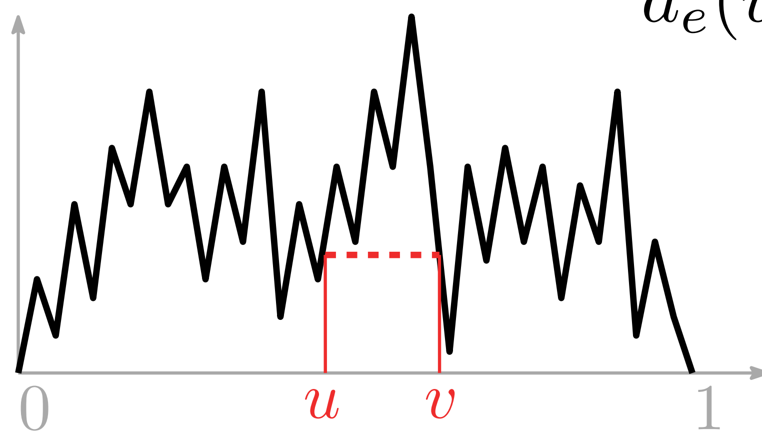
Conditional on $\mathcal{T}_e$, Z a centered Gaussian process with $Z_\rho = 0$ and $E[(Z_s - Z_t)^2] = d_e(s,t)$

$Z \sim$ Brownian motion on the tree

Brownian snake $(e_t, Z_t)_{0\leq t\leq 1}$

1st step : the Brownian tree [Aldous]





$d_e(u,v) = e_u + e_v - 2\min_{u\leq s\leq v} e_s$

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$Z \sim$ **Brownian motion on the tree**

Theorem :

$$\left((8n)^{-1/2} C_{\lfloor nt\rfloor}, (2n/1)^{-1/4}\tilde{Z}_{\lfloor nt\rfloor}\right)_{0\leq t\leq 1} \xrightarrow[n\to\infty]{(d)} (e_t, Z_t)_{0\leq t\leq 1},$$

Distances in simple outer-triangular maps

To do that :

- encode the maps by some trees.
- study the limits of trees,
- **interpret the distance in the maps by some function of the tree**.

S_n = outer-triangular simple map
$(C_{\lfloor nt \rfloor}, \tilde{Z}_{\lfloor nt \rfloor})$ = contour and label process of the associated tree
$Z_{\lfloor nt \rfloor}$ = distance **in the map** between vertex "$\lfloor nt \rfloor$" and the root.

Theorem :
S_n= random outer-triangular simple map, then for all $\varepsilon > 0$:

$$\mathbb{P}\left(\sup_{0 \le t \le 1} \left\{ \left| \tilde{Z}_{\lfloor nt \rfloor} - Z_{\lfloor nt \rfloor} \right| \right\} \ge \varepsilon n^{1/4} \right) \to 0.$$

i.e. the label process of the tree gives the distance to the root in the map.

Distances in simple maps

Claim : $d_M(root, u) \leq$ Label of u

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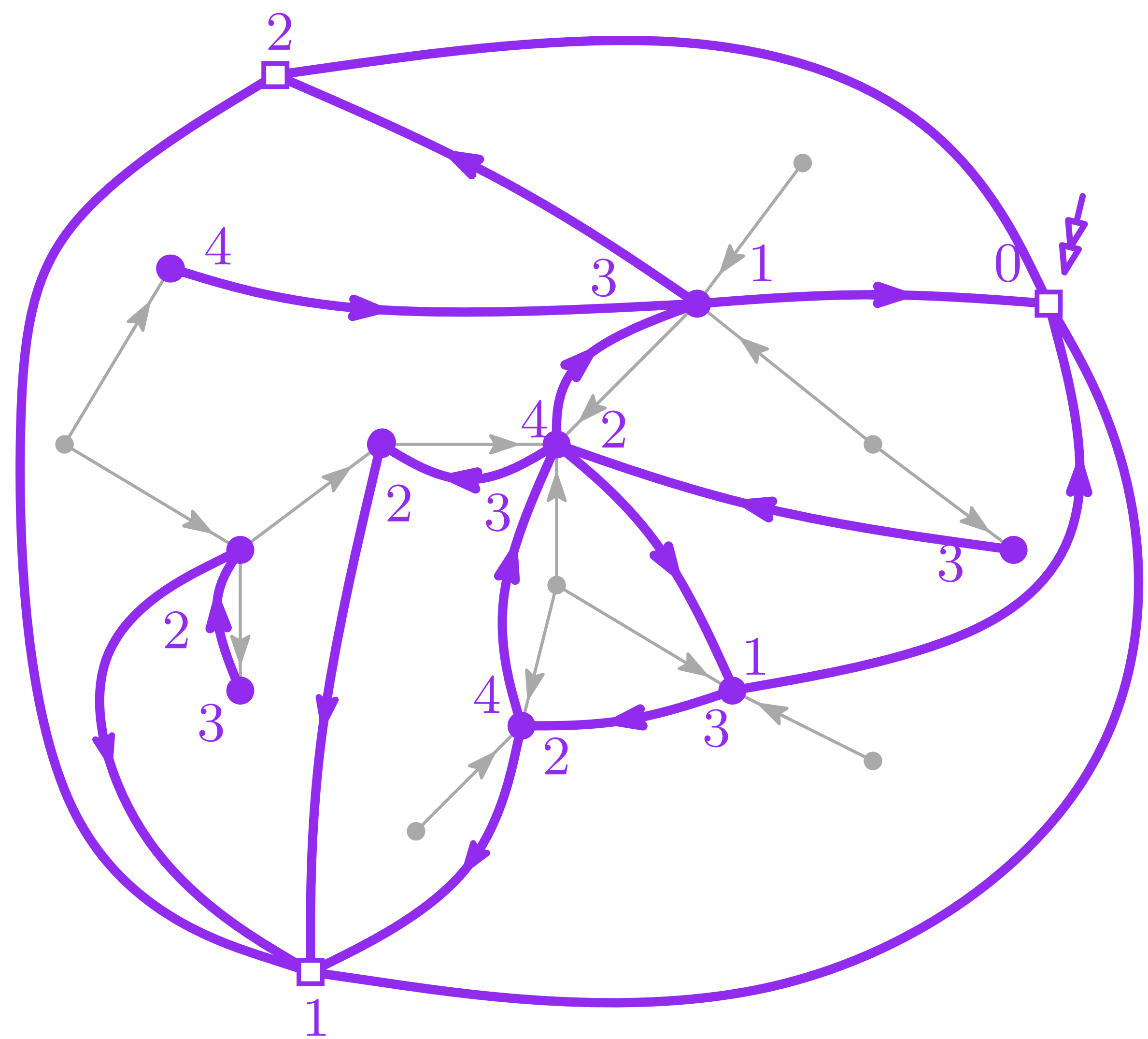
- Consider the **Left Most Path** from (u, v) to the root face.



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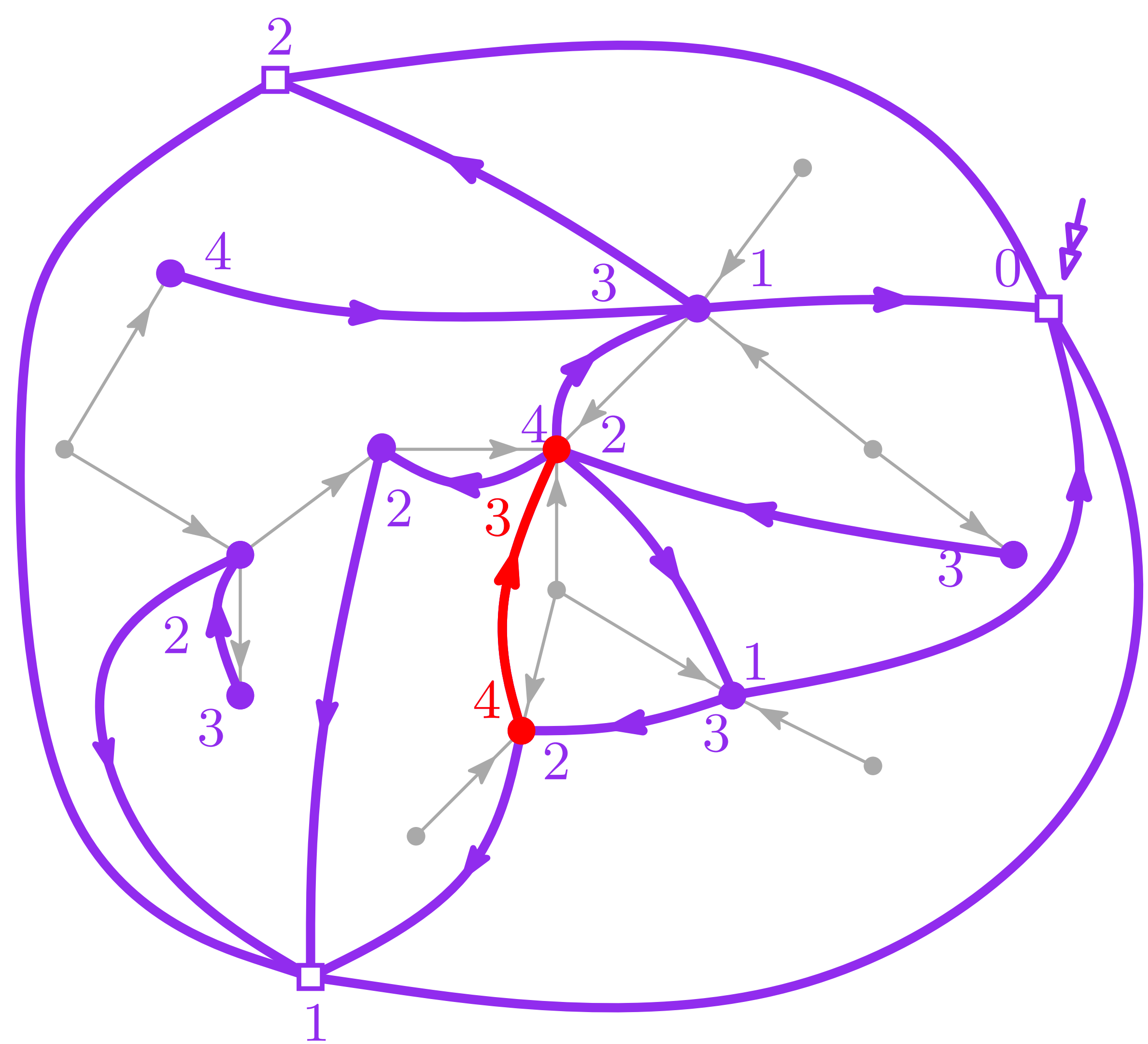

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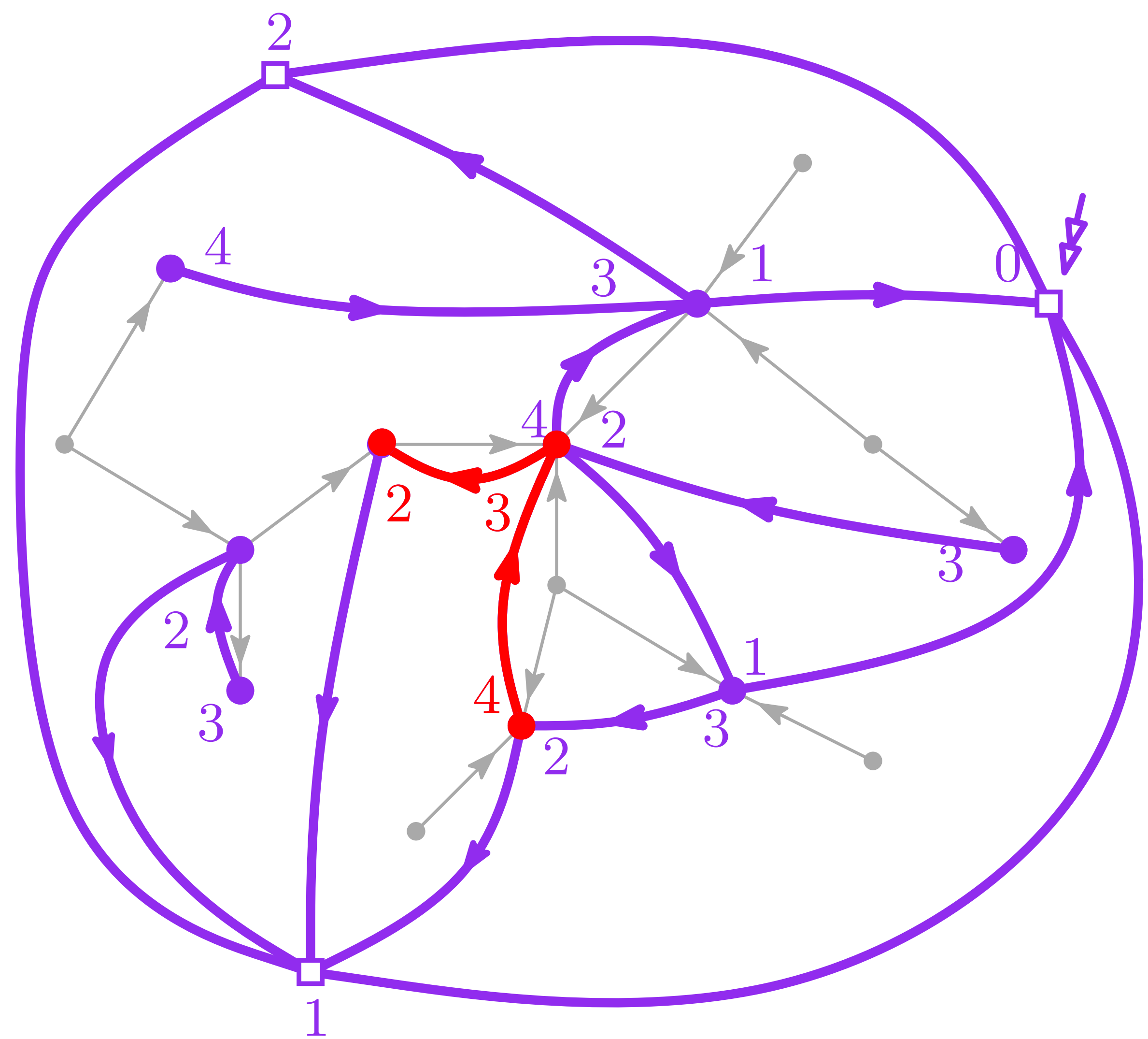
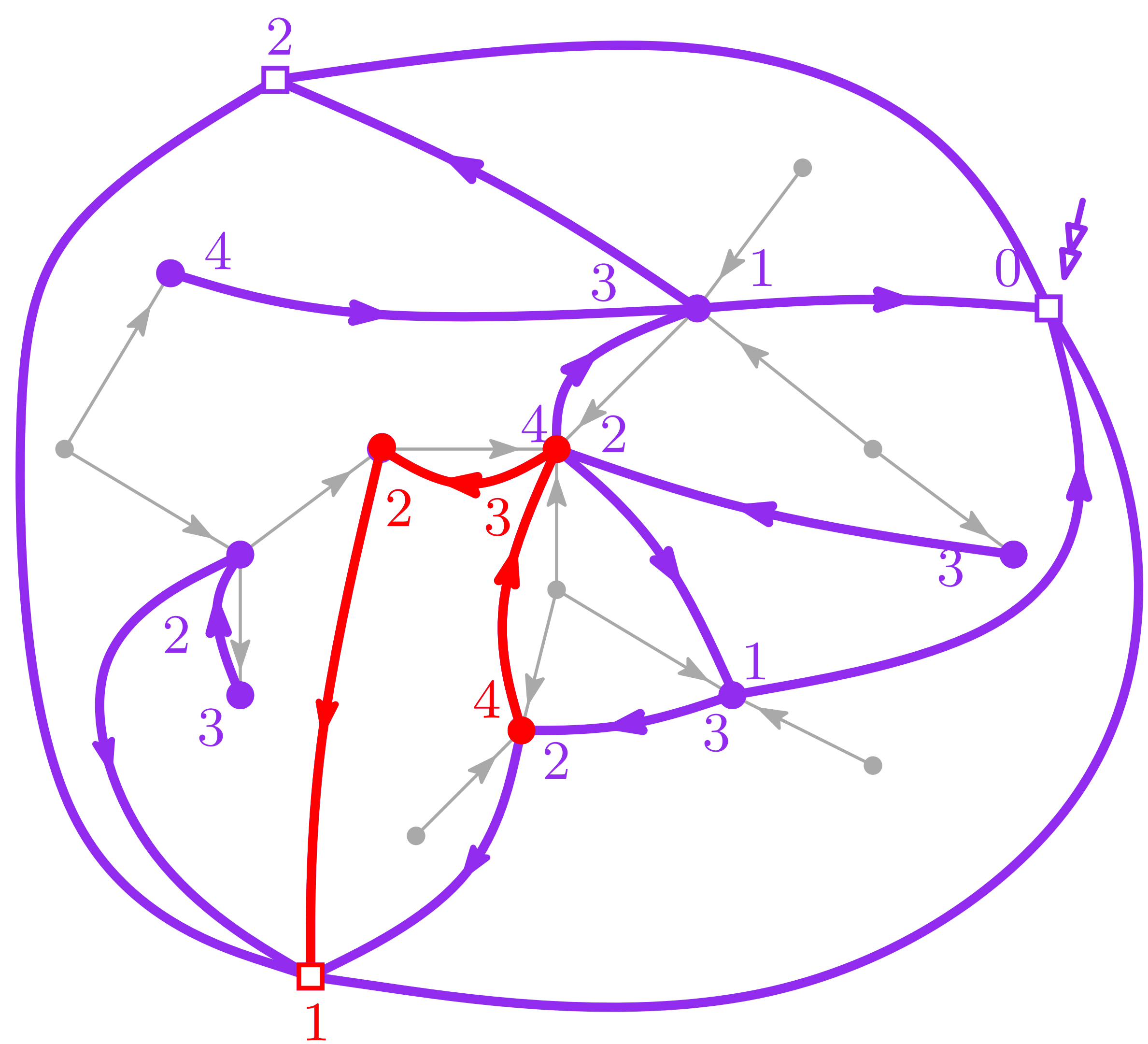


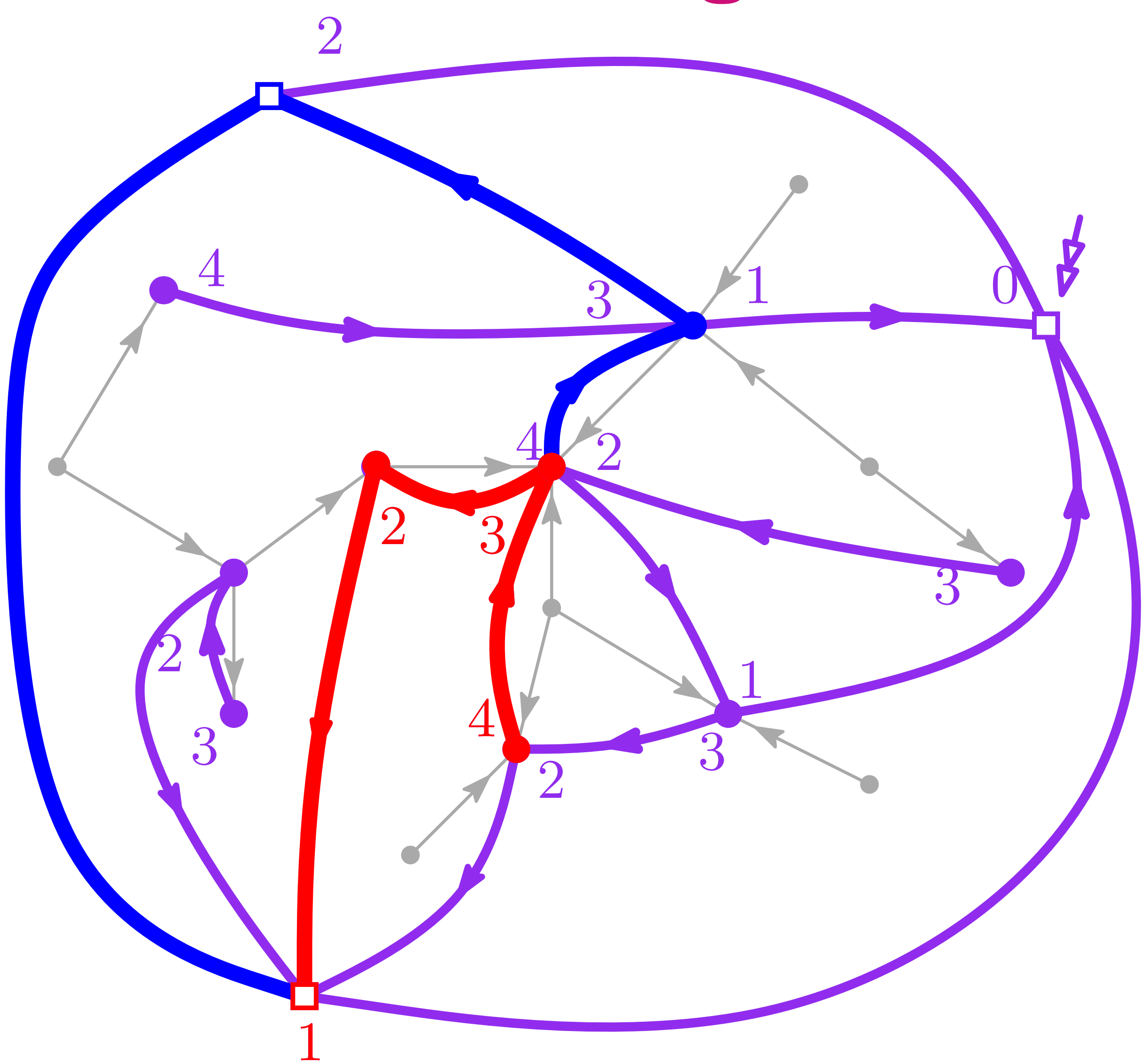
Distances in simple maps

Claim : $d_M(root, u) \leq$ Label of u

- Consider the **Left Most Path** from (u, v) to the root face.
- From the property of the closure, on the left of the LMP the labels decrease exactly by 1.
- The LMP is not self-intersecting: it reaches the outer-face



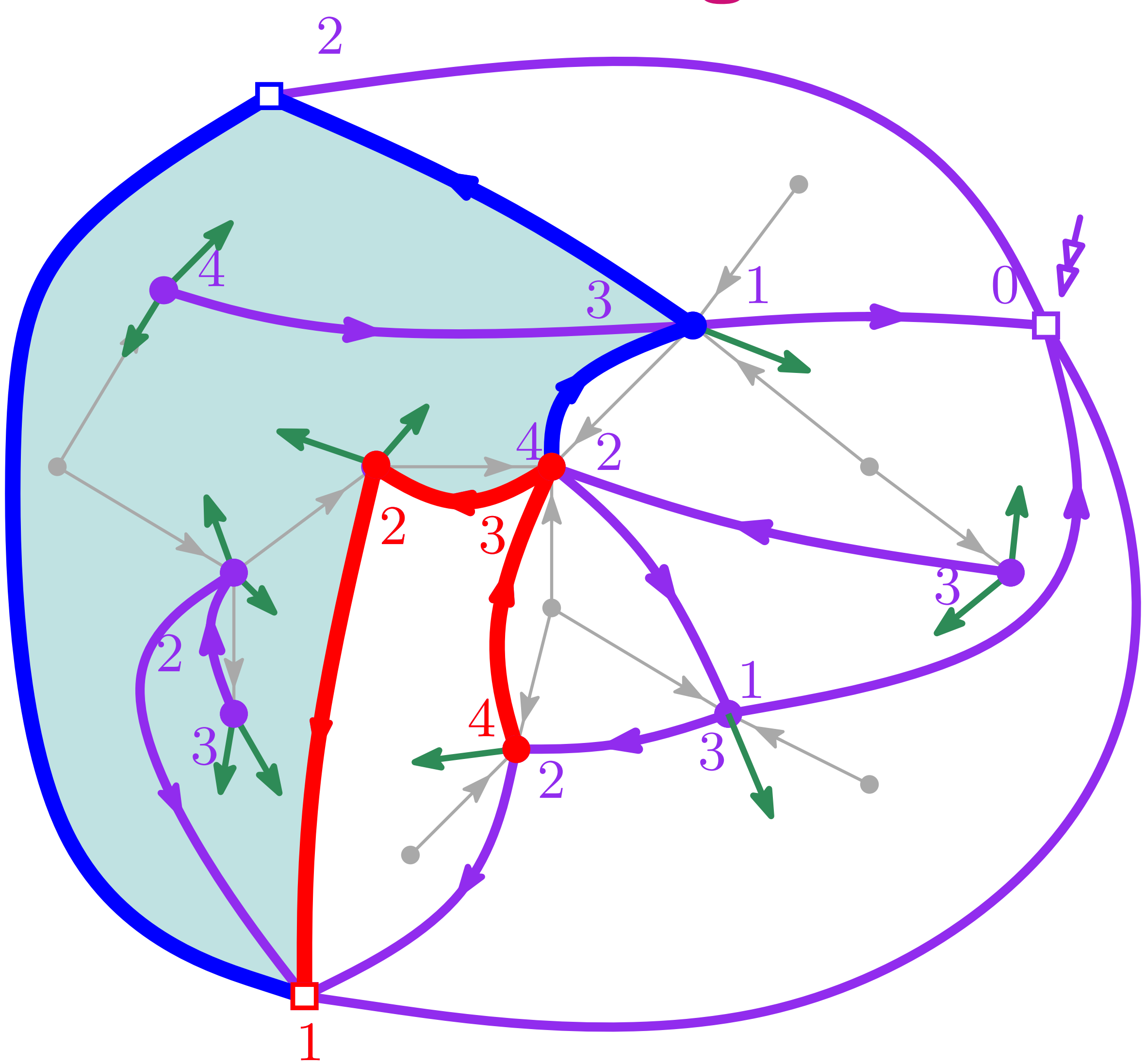
LMP are almost geodesic



Leftmost path

Another path: can it be shorter ?

LMP are almost geodesic



Consider the 3-orientation of the map with buds

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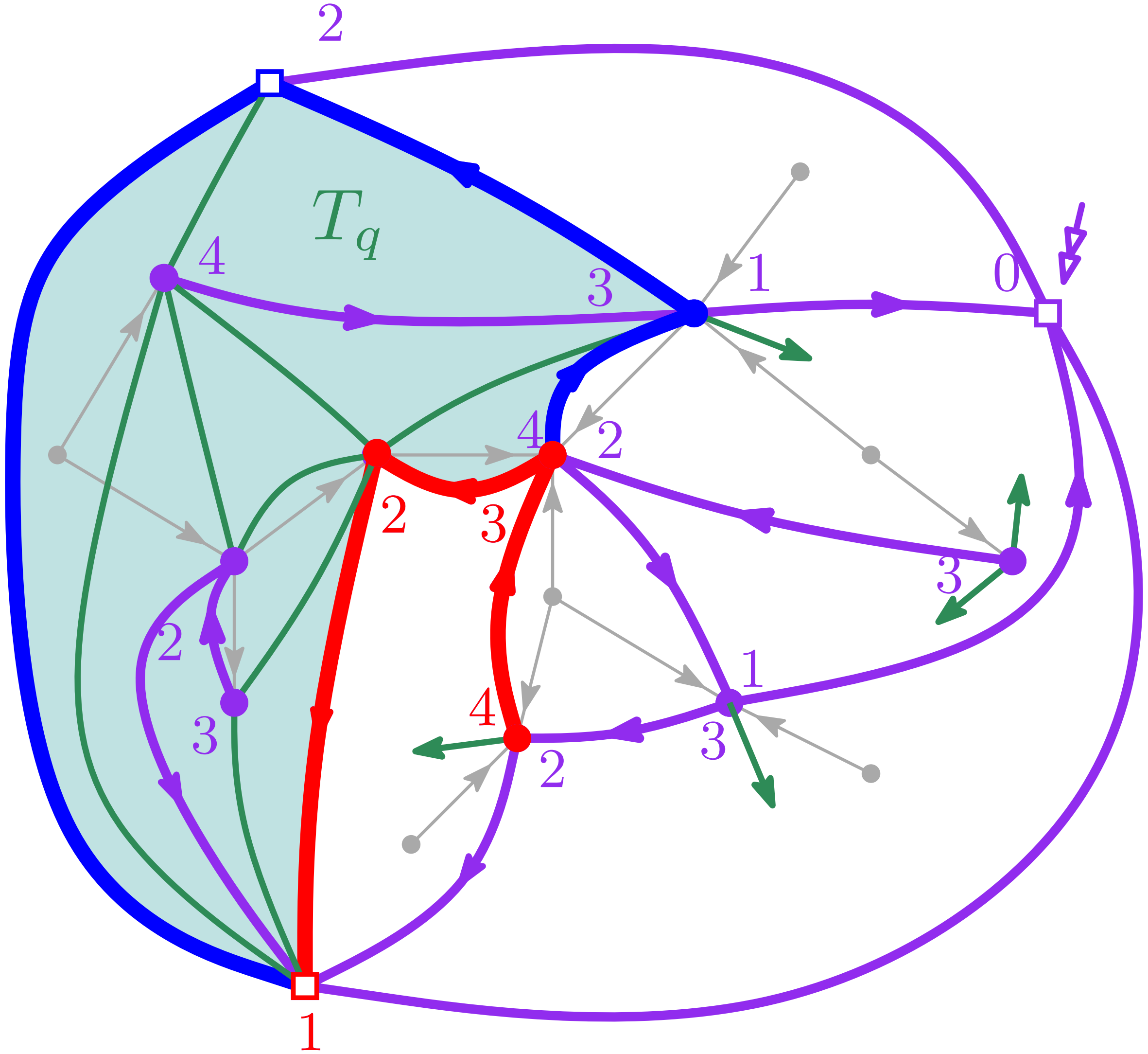

Consider the 3-orientation of the map with buds

Use the buds to triangulate the submap surrounded by the two paths.

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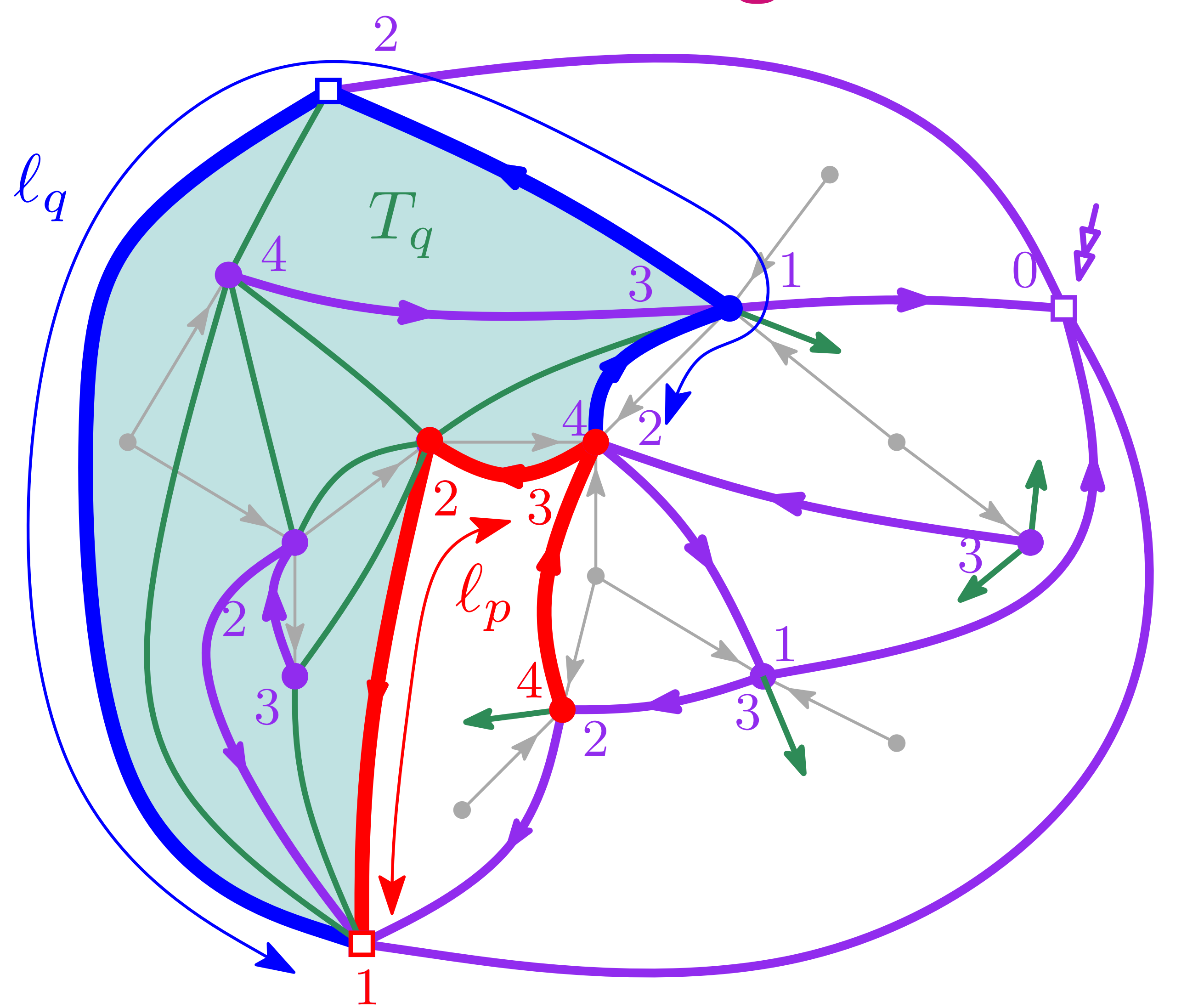


Leftmost path

Another path: can it be shorter ?

Consider the 3-orientation of the map with buds

Use the buds to triangulate the submap surrounded by the two paths.

Euler Formula :

$$|E(T_q)| = 3|V(T_q)| - 3 - (\ell_p + \ell_q)$$

3-orientation + LMP :

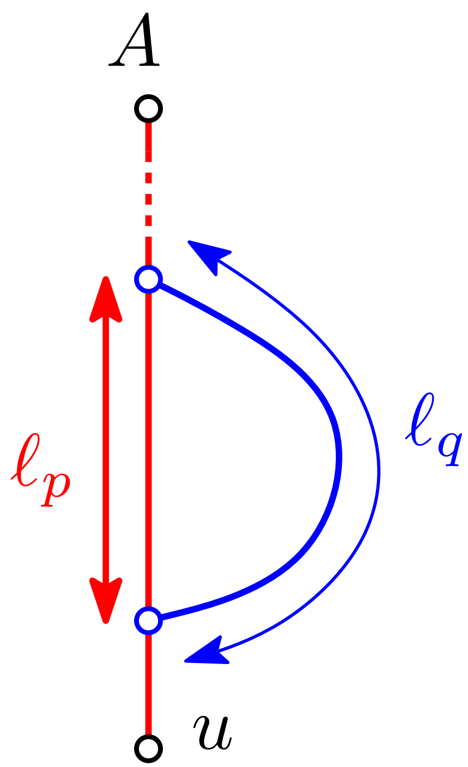
$$|E(T_q)| \geq 3|V(T_q)| - 2\ell_q - 2$$

$$\Longrightarrow \ell_q \geq \ell_p + 1$$

LMP are almost geodesic

Leftmost path

Another path: can it be shorter ?

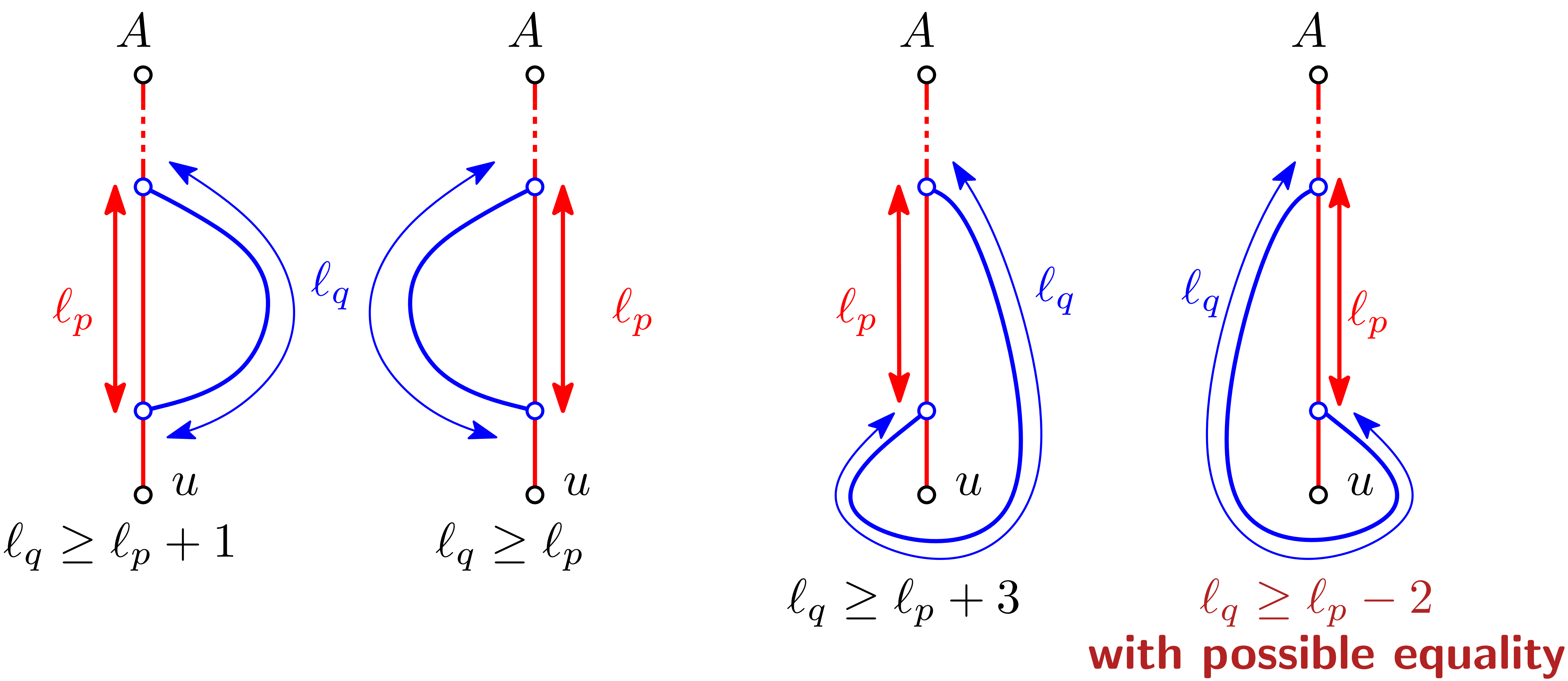


$$\ell_q \geq \ell_p + 1$$

LMP are almost geodesic

Leftmost path

Another path: can it be shorter ? YES



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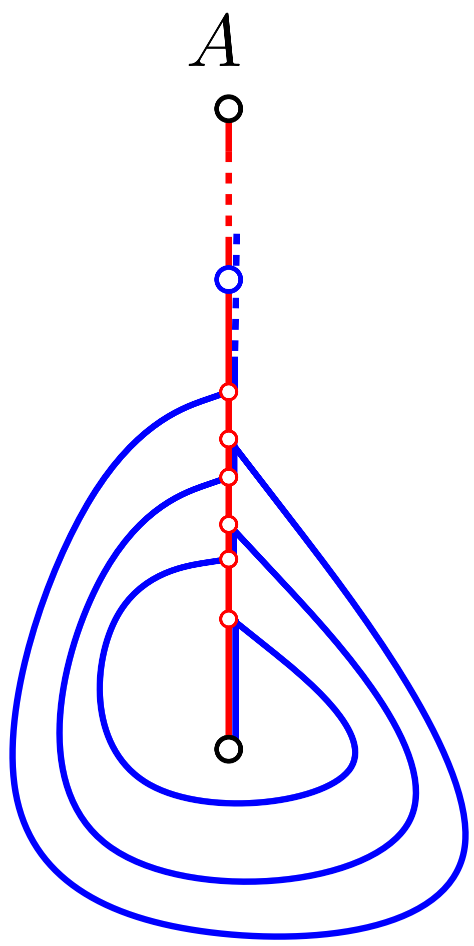
Another path: can it be shorter ? YES ... but not too often



Bad configuration =
too many **windings** around the LMP

But w.h.p a winding cannot be too short.

$\Longrightarrow$ w.h.p the number of windings is $o(n^{1/4})$.

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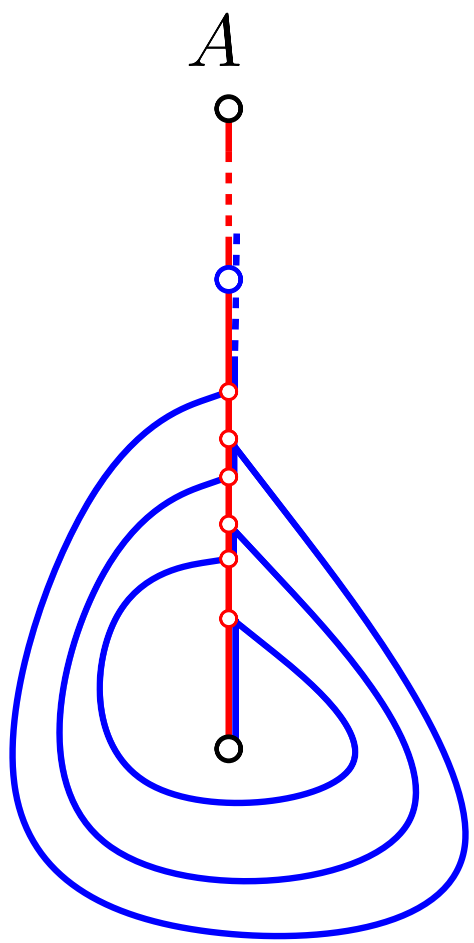

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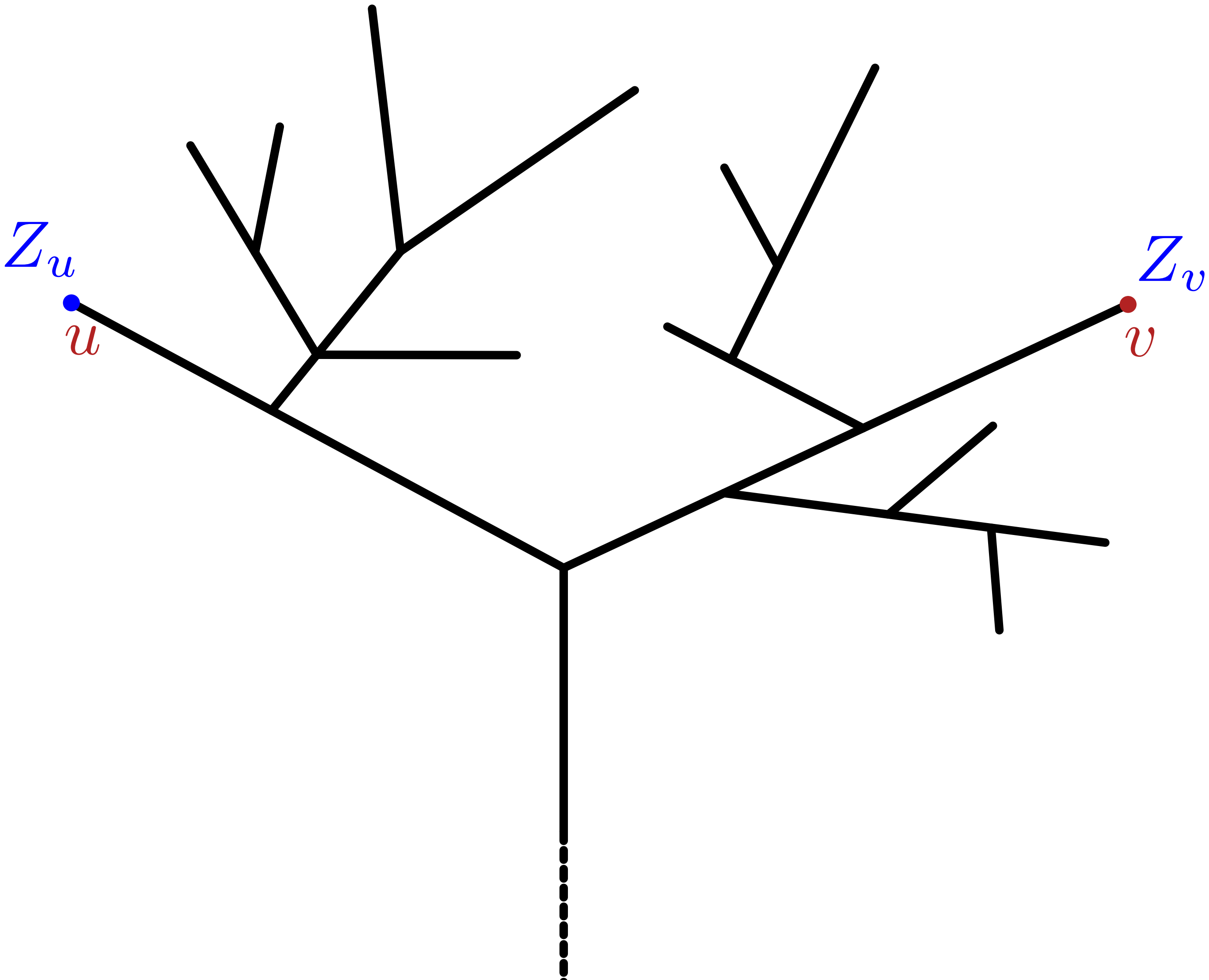
Proposition: [Addario-Berry, A. '13]
For $\varepsilon > 0$, let $A_{n,\varepsilon}$ be the event that there exists $u \in M_n$ such that
Label of $u \geq d_{M_n}(u, root) + \varepsilon n^{1/4}$.
Then under the uniform law on $\mathcal{M}_n$, for all $\varepsilon > 0$:

$$\mathbb{P}\left(A_{n,\varepsilon}\right) \to 0.$$

Distances are tight



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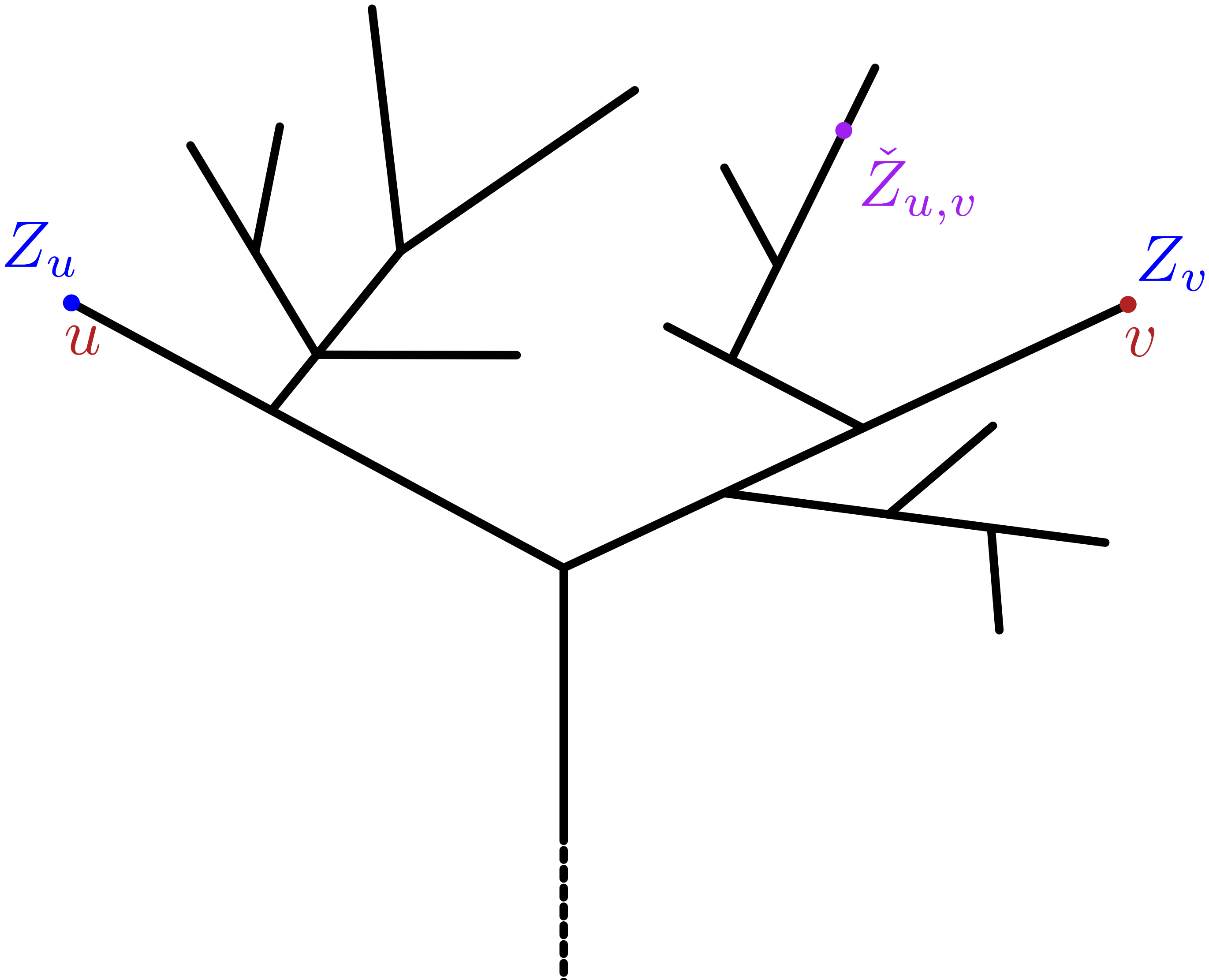
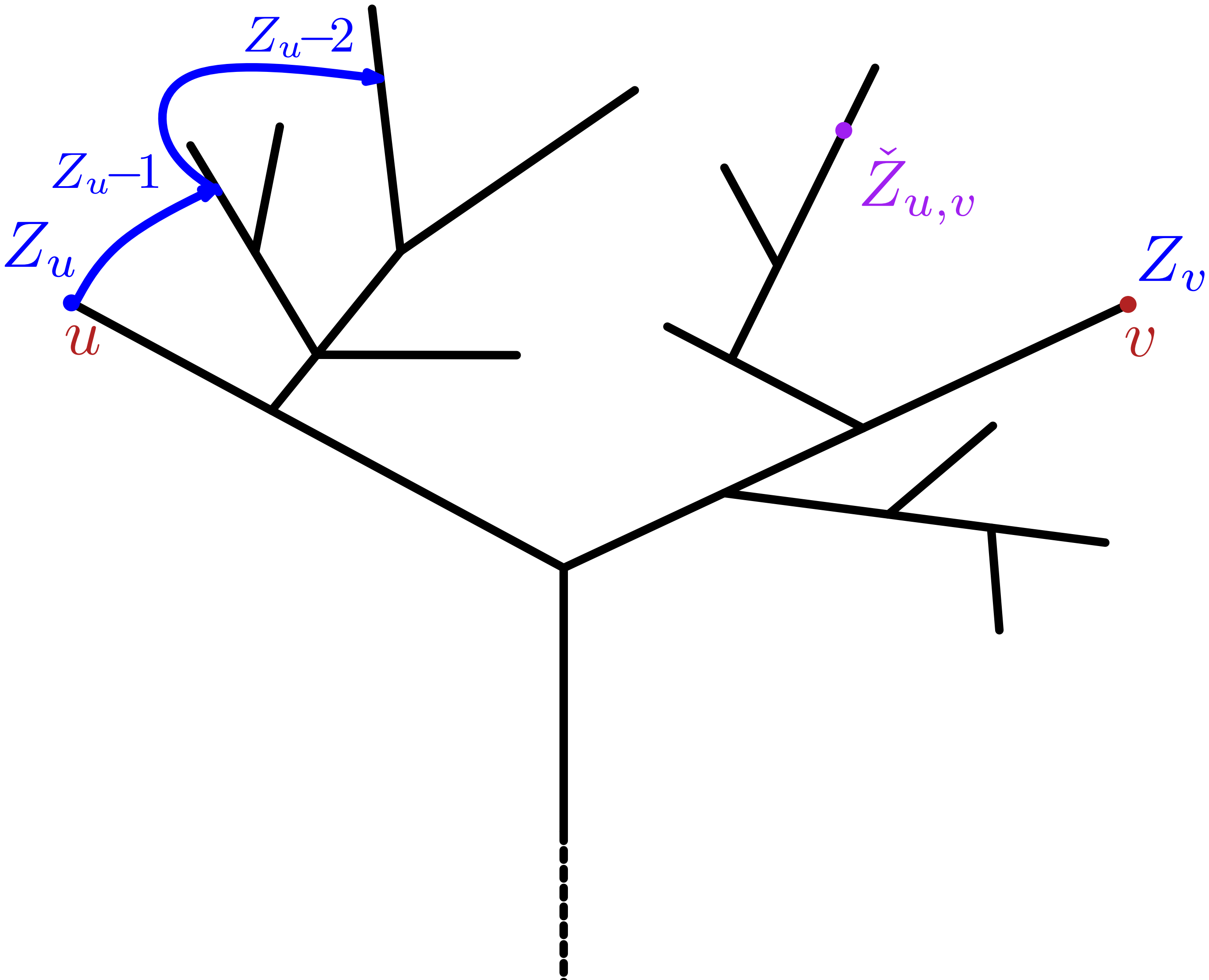


$$\check{Z}_{u,v} = \min\{Z_s, u \leq s \leq v\}$$

Distances are tight



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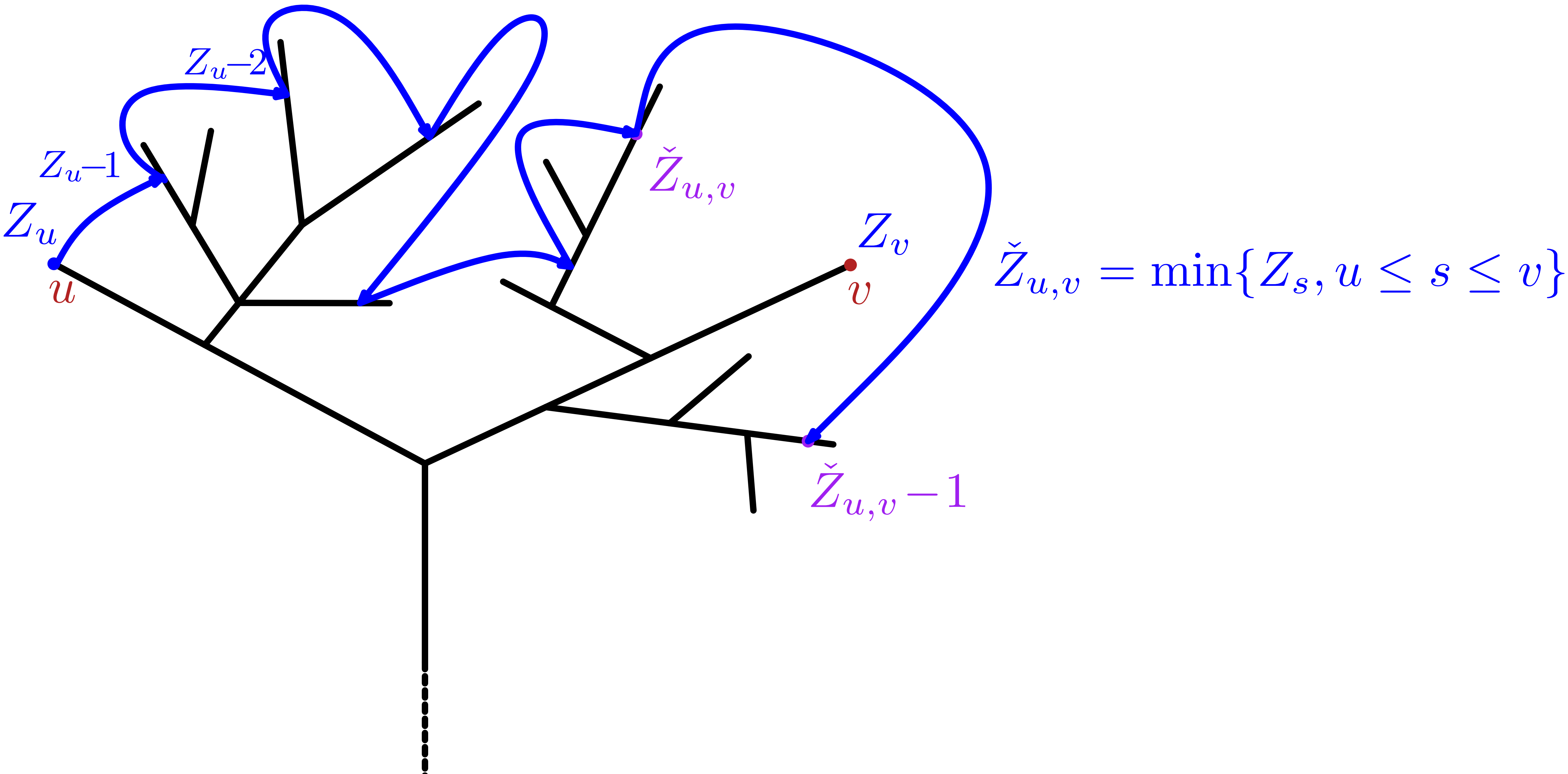

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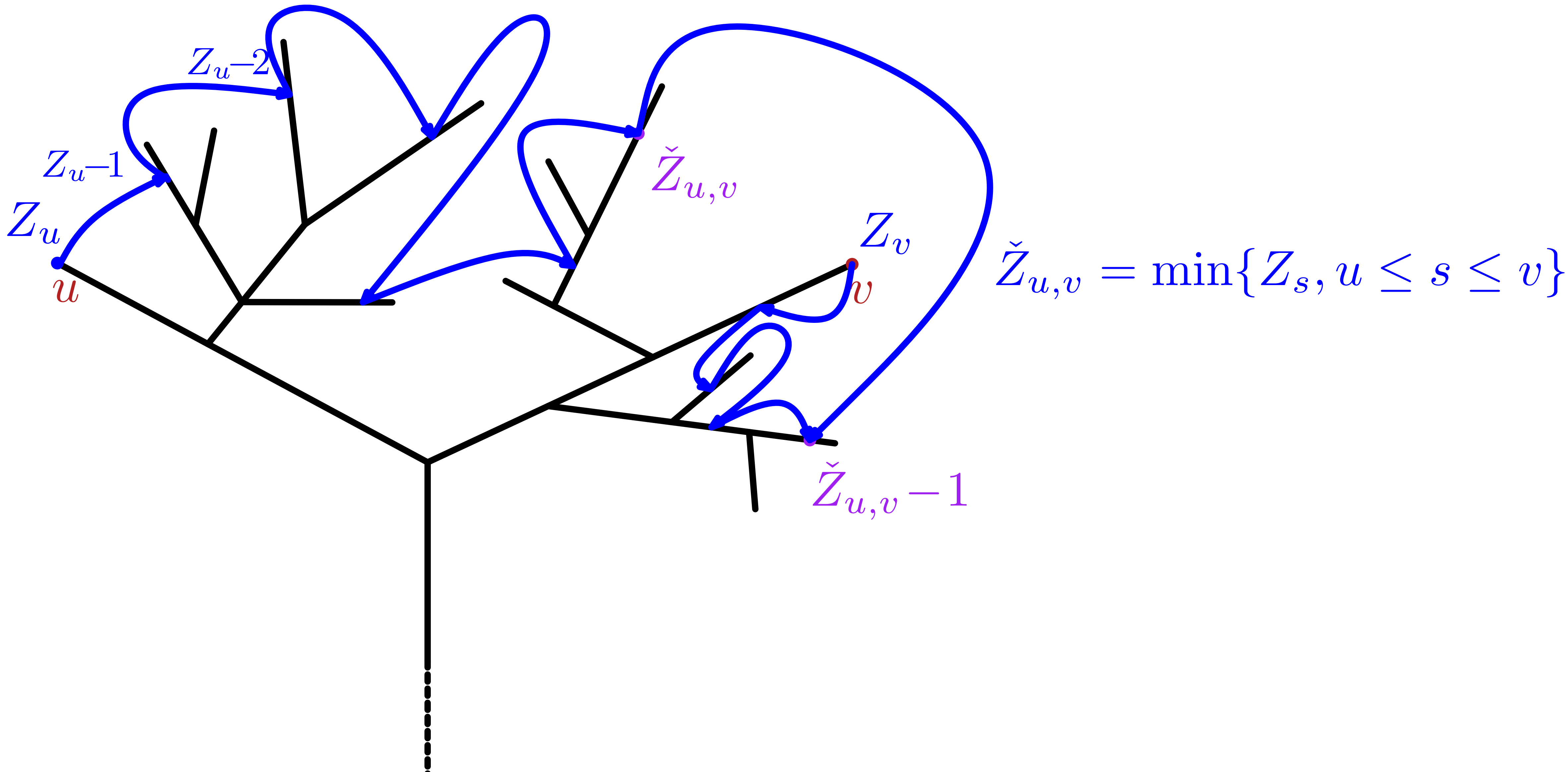

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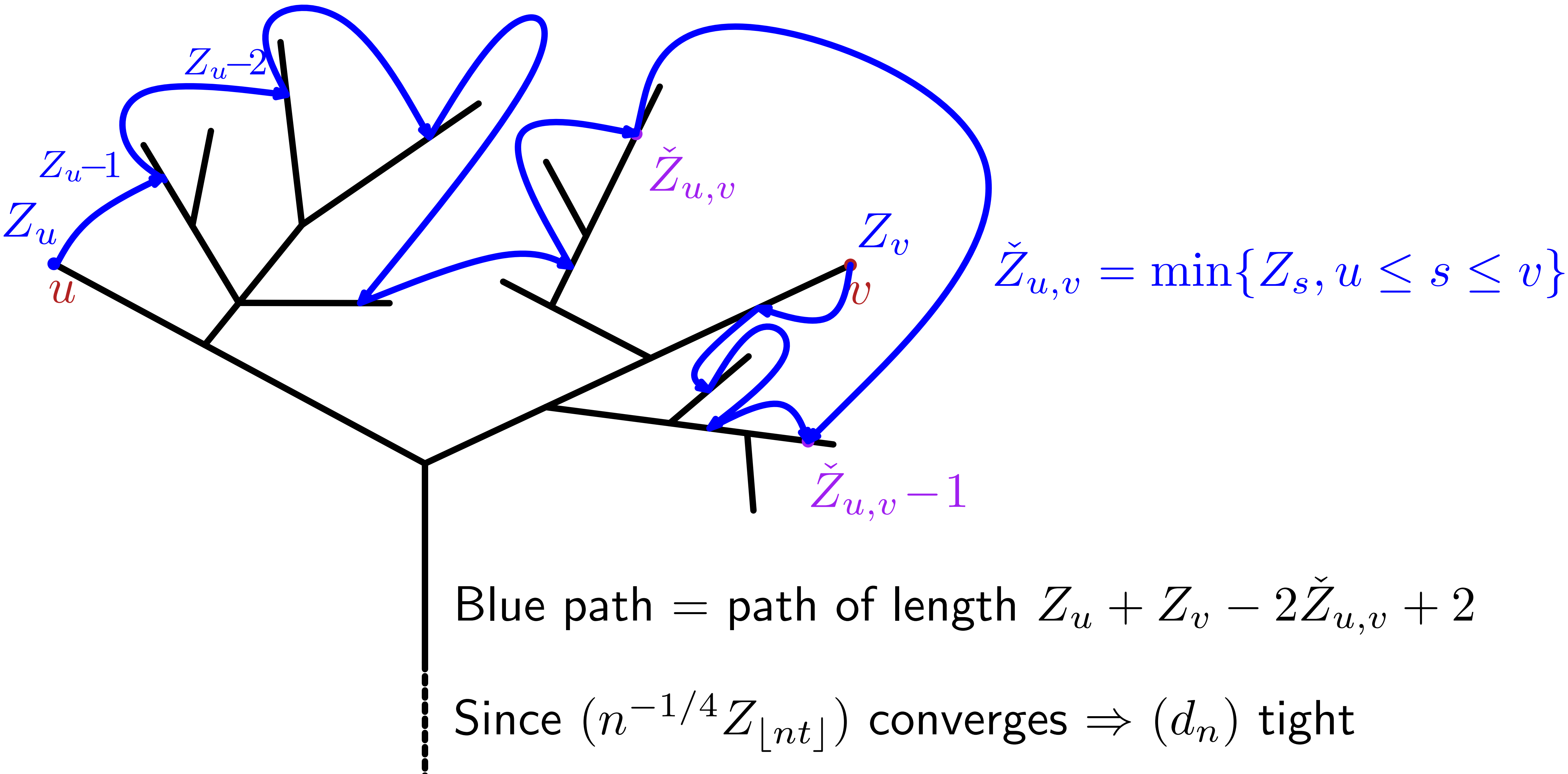


Blue path = path of length $Z_u + Z_v - 2\check{Z}_{u,v} + 2$

Since $(n^{-1/4} Z_{\lfloor nt \rfloor})$ converges $\Rightarrow$ (d_n) tight

The result for the last time

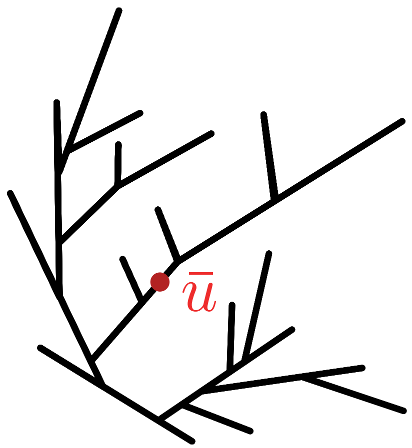
Theorem : [A., Bernardi, Collet, Fusy]
$\mathcal{S}_n = \{$ simple maps with n edges $\}$
$S_n =$ uniform random element of $\mathcal{S}_n$. Then:

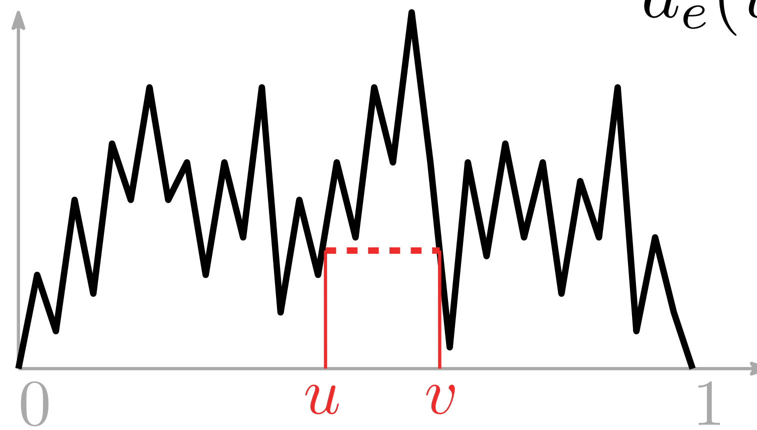
$$\left(S_n, \left(\frac{1}{2n}\right)^{1/4} d_{S_n}\right) \xrightarrow{(d)} (M, D^\star),$$

for the distance of Gromov-Hausdorff on the isometry classes of compact metric spaces.

The Brownian Map ??

The Brownian map





$d_e(u,v) = e_u + e_v - 2\min_{u \le s \le v} e_s$

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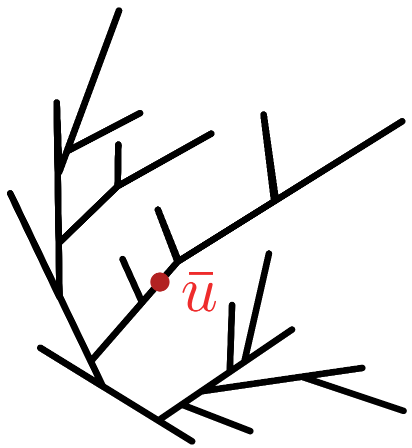

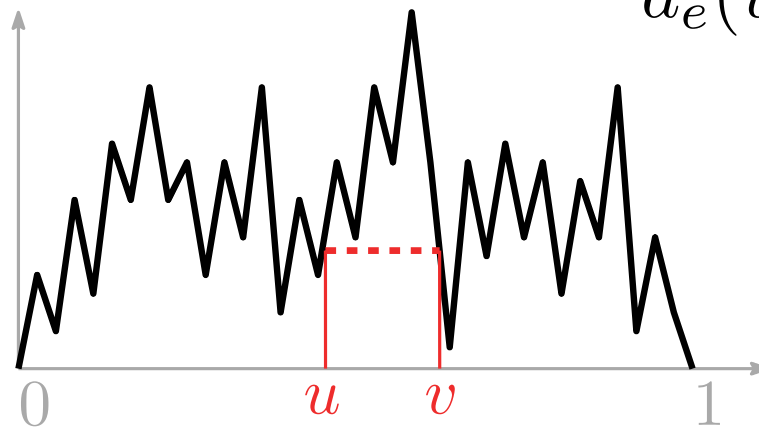

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$$D^\circ(s,t) = Z_s + Z_t - 2\max\Big(\inf_{s\le u\le t} Z_u, \inf_{t\le u\le s} Z_u\Big), \quad s,t \in [0,1].$$

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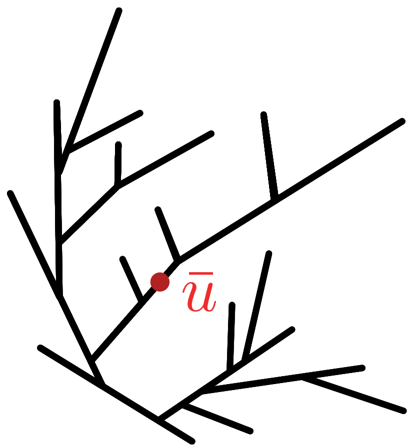

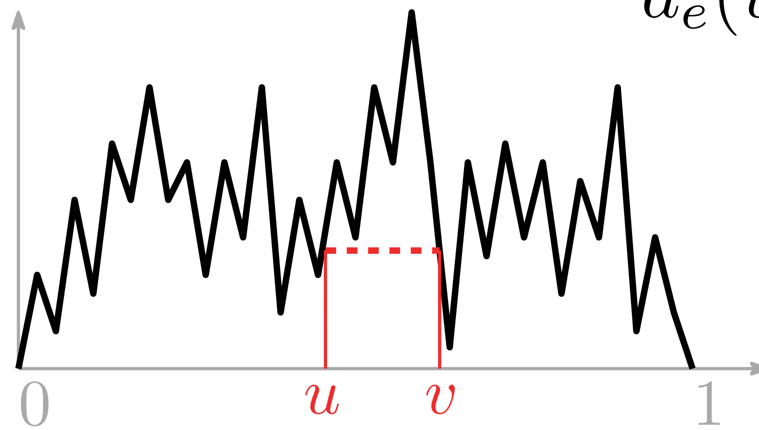

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$$D^*(a,b) = \inf\left\{\sum_{i=1}^{k-1} D^\circ(a_i, a_{i+1}) : k \ge 1, a = a_1, a_2, \ldots, a_{k-1}, a_k = b\right\},$$

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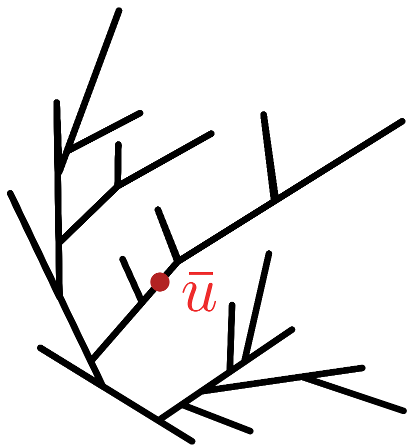

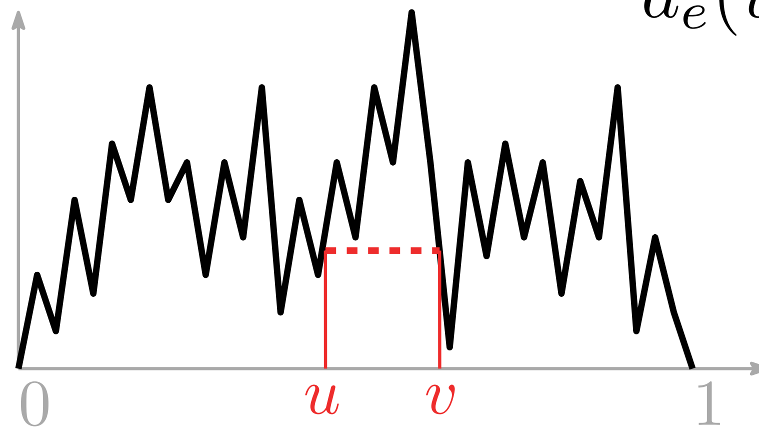

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Then $M = (\mathcal{T}_e/\sim_{D^*}, D^*)$ is the **Brownian map**.

A word of conclusion

Nice to see that the idea of LMP introduced for simple triangulations also work for simple maps.

Natural further step: try to adapt the techniques for all the bijections involving blossoming trees. In particular in the unified setting of [Bernardi, Fusy '10] and [A., Poulalhon '14]

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Question: Can we make it work for 3-connected planar maps ?

By Whitney's theorem, a 3-connected planar **graph** has a unique embedding as a planar **map**.

$\Rightarrow$ would permit to get results about 3-connected planar graphs (and then about 2-connected and connected planar graphs).

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